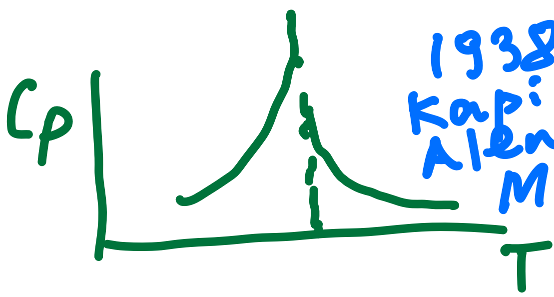
Gentle intro to quantum turbulence

QT is realised in superfluids

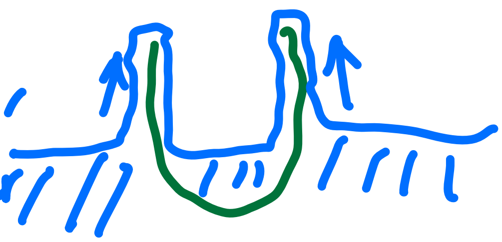
Examples: • ^4He at $T < T_\lambda = 2.17K$

Keesom & Keesom '1935

Daunt & Mendelssohn '38



1938 Kapitsa, Alen, Misener



creeping effect.

• ^3He at $T < T_c = 2.49 \cdot 10^{-3}$ K

• Bose-Enstein condensate at $T \sim 100$ nK

• Neutron stars, super-heavy atomic nuclei

• magnons, polaritons, excitons.

• Superfluid vacuum theory SVT to unite Standard Model and Gravity.

Overview $\boxed{\text{Mostly } T \simeq 0}$

- QT of quantized vortices
  (i) gas of point vortices in 2D
  (ii) vortex tangles in 3D
- Small-scale QT as a cascade of interacting Kelvin waves
- Reconnections and bottlenecks at transient (intervortex) scales
- Wave turbulence
  (i) 4-wave mixing of matter waves
  (ii) 3-wave mixing of Bogolubov waves (acoustic + short w).
  (iv) 4-wave process of Kelvin waves

Unifying element: emergence of long range order described by a complex field ψ. Bose condensation.

What is quantum turbulence?

Turbulence is a chaotic motion of a fluid. Quantum mechanics is about evolution of a probability given by $|\psi|^2$ where ψ is a complex field. Can we view this field as a fluid?

The answer is yes, it was given by Madelung in 1926 — the same year as the publication by Shrödinger of his famous equation

$$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \qquad (*)$$

The fluid representation work even for

a single particle!

Its clear that the fluid density should be $\boxed{\rho = m|\psi|^2}$ (since this is the density of probability).

But what is fluid velocity? we know that the QM momentum operator is $\hat{p} = -i\hbar\nabla$

So $\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle =$

$$-i\hbar\int\psi^*\nabla\psi\,d\bar{x} = -i\frac{\hbar}{2}\int(\psi^*\nabla\psi - \psi\nabla\psi^*)\,d\bar{x}$$

$$= \frac{\hbar}{2m}\int\rho\nabla\varphi\,d\bar{x}$$ where $\boxed{\psi = \sqrt{\rho/m}\,e^{i\varphi}}$

since the momentum density is $\vec{P} = \rho\vec{u}/2$ we have for the velocity field $\boxed{\vec{u} = \frac{\hbar}{m}\nabla\varphi}$

Do such ρ and $\bar{u}$ satisfy fluid equations?

$$i\hbar\,\partial_t|\psi|^2 = -\frac{\hbar^2}{2m}\underbrace{\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)}$$

$$\underbrace{\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)}$$

$$\psi = |\psi|e^{i\varphi}$$

$$\underbrace{\nabla\cdot\left(\psi^*\psi\left(\frac{\nabla\psi}{\psi} - \frac{\nabla\psi^*}{\psi^*}\right)\right)}$$

$$\frac{1}{m}\nabla\cdot\left(\rho\, 2i\nabla\varphi\right)$$

$$\Rightarrow \boxed{\partial_t\rho = -\nabla\cdot(\rho\vec{u})}$$

This is the mass balance equation.

Now let us consider the momentum balance.

$$2i\dot{\varphi} = \left(\frac{\partial_t \psi}{\psi} - \frac{\partial_t \psi^*}{\psi^*}\right) =$$

$$i\frac{\hbar}{2m}\left(\frac{\nabla^2\psi}{\psi} + \frac{\nabla^2\psi^*}{\psi^*}\right)$$

$$\sqrt{m}\,\nabla^2\psi = \nabla\cdot\left((\nabla\sqrt{\rho})e^{i\varphi} + \sqrt{\rho}\,i(\nabla\varphi)\,e^{i\varphi}\right)$$

$$= \nabla^2\sqrt{\rho}\,e^{i\varphi} + 2(\nabla\sqrt{\rho})\,i(\nabla\varphi)\,e^{i\varphi} + i\nabla^2\varphi\,e^{i\varphi}$$

$$-\sqrt{\rho}\,(\nabla\varphi)^2 e^{i\varphi}$$

$$\Rightarrow \quad 2\dot{\varphi} = \frac{\hbar}{2m}\left(2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} - 2(\nabla\varphi)^2\right)$$

This is a Bernoulli equation.

Taking ∇:

$$\boxed{\partial_t\bar{u} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{1}{m}\nabla Q, \quad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}}$$

This is the momentum balance eq.
p is usually called quantum pressure, but more precisely it is q. enthalpy (g-factor) a.k.a q. potential (Bohm).
In fact, pressure in this case is a tensor:

exersize

$$\overleftrightarrow{P}_Q = -\left(\frac{\hbar}{2m}\right)^2 \rho \nabla \otimes \nabla \ln \rho$$

$$\nabla Q = \frac{m}{\rho} \nabla \cdot \overleftrightarrow{P}_Q$$

Summarising, after Madelung, the 1-particle Schrödinger eq. become fluid equations:

$$\boxed{\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \bar{u}) = 0, \\ &\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{\nabla \cdot \overleftrightarrow{P}_Q}{\rho}. \end{aligned}}$$

So, even on the 1 particle level, quantum motion is similar to a fluid flow.

Recall wave-particle duality

Now we see

fluid-particle duality.

But what is exactly quantised in (*)? E is quantised if particle is trapped by a $U(\vec{x})$, but it has a continuous spectrum without U:

$$\Psi_E \sim e^{-i\frac{E}{\hbar}t + i\vec{k}\cdot\vec{x}} \qquad \vec{k} \in \mathbb{R}^d$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad (\vec{p} = \hbar\vec{k})$$

These are De Broglie matter waves.

It is circulation that must be quantised! Onsager '49 Feynmann '55

<u>Circulation</u>: $\Gamma = \oint_C \vec{u} \cdot d\vec{s}$

But the flow is irrotational

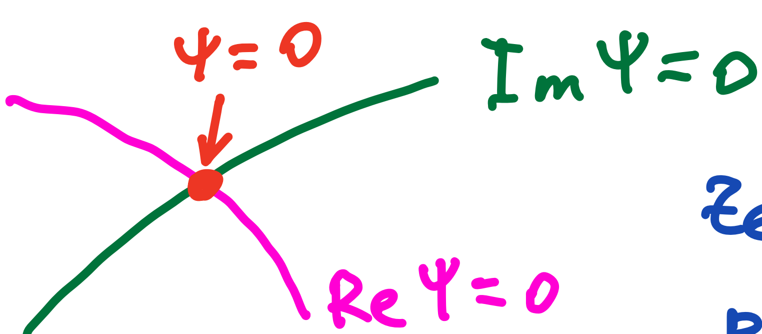
$$\vec{u} = \frac{\hbar}{m} \nabla \varphi$$

everywhere where ψ is differentiable

so $\Gamma = 0$ if φ is defined and differentiable inside contour C.

Since $\psi = \sqrt{\rho_m}\, e^{i\varphi}$, φ is undefined if $\rho = 0$. Consider a generic smooth ψ that has a zero:

in 2D:



Zeros are points.

In 3D – lines (intersection of surfaces $\mathrm{Re}\,\Psi = 0$ and $\mathrm{Im}\,\Psi = 0$).

Locally near 0 in 2D:

$$\mathrm{Re}\,\Psi = a(x-x_0) + b(y-y_0)$$

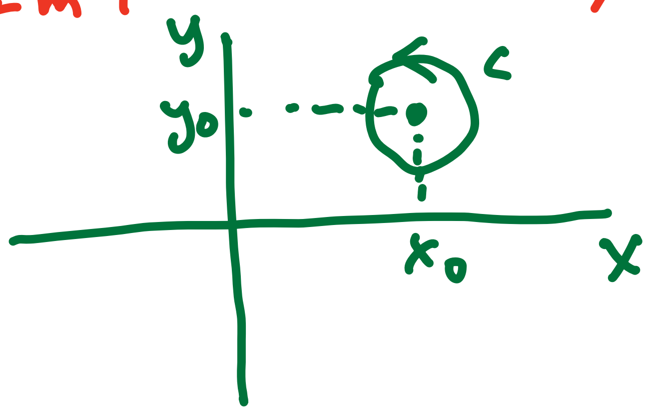
$$\mathrm{Im}\,\Psi = c(x-x_0) + d(y-y_0)$$

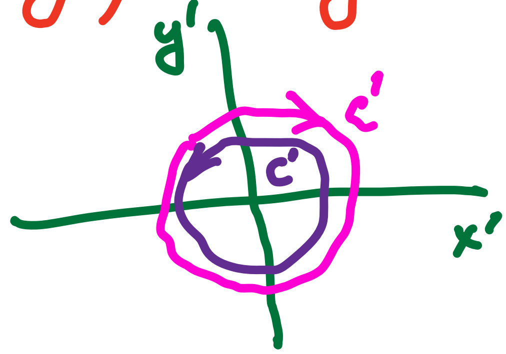
Note that this is a stationary solution of $(*)$.

Introduce local coords $\tilde{x}, \tilde{y}$

$$\mathrm{Re}\,\Psi = a(x-x_0) + b(y-y_0) = \tilde{x}$$

$$\mathrm{Im}\,\Psi = c(x-x_0) + d(y-y_0) = \tilde{y}$$





In polar coordinates $(\tilde{x}, \tilde{y}) = r e^{i\theta}$:

$$\Psi = \tilde{x} + i\tilde{y} = r e^{i\theta} \Rightarrow \varphi = \theta$$

$\Psi = 0 \rightarrow r = 0$

$$\Gamma = \oint_c \vec{u} \cdot d\vec{s} = \frac{\hbar}{m} \int_c \nabla \varphi \cdot d\vec{s} = \frac{\hbar}{m} \int_{c'} \nabla \theta \cdot d\vec{s} = \pm \frac{2\pi\hbar}{m}$$

$$2\pi\hbar = h$$

$\Rightarrow \Gamma = \pm \frac{h}{m}$ ← Quantum of circulation.

We considered simple zeros.

For n-th order 0: $\Gamma = \pm \frac{hn}{m}$

Exercize: proof this for

$$\mathrm{Re}\,\Psi = x^n, \quad \mathrm{Im}\,\Psi = y^n.$$

Note: Small smooth perturbation of the n-th order 0 splits it to n 1-st order 0's.

$\Rightarrow$ Multiply charged vortex is unstable.

So even 1-particle linear QM exhibits fluid properties and the vortices are quantized. (points in 2D, lines in 3D)

But of course there is no turbulence in this linear system: General solution is an arbitrary linear superposition of the matter waves. Vortices, zeros of such a wave field, do not have an individual dynamical role.

BEC is a collective phenomenon where interaction of particles is important. A minimal model of BEC was derived for a dilute gas of bosonic atoms (^{87}Rb, ^{23}Na, ^{7}Li).

This is Gross-Pitaevskii equation

$$i\hbar\,\partial_t \psi = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi$$

interaction term

$g > 0$ repulsive interaction (^{87}Rb, ^{23}Na)

$g < 0$ attractive interaction (^{7}Li).

Note ^4He is a liquid, not a dilute gas $\Rightarrow$ GPE is not applicable.

However, most basic properties

of superfluid ^4He (i.e. He II) at $T \simeq 0$ could be understood from the GPE. At $T > 0$, He II can be described as a mixture of superfluid and a normal (viscous) fluid.

→ Tisza-Landau 2-fluid model vorticity in which is not quantised

It is remarkable that GPE can be viewed as a "microscopic" model, coarse graining which one gets a 2F model:

- Coarse grained quantum vortices → superfluid
- Coarse grained phonons (sound waves) → normal fluid.
- Scattering of sound by vortices - mutual friction.

But I am not aware of works deriving 2F from GPE by coarse graining.

nice project!

First, let us consider a <u>uniform density state</u>.

It corresponds to GPE solution

$$\psi = A_0 e^{-\frac{g}{\hbar} A_0^2 t}, \quad A_0 = \text{const}.$$

$$\rho = \rho_0 = m A_0^2 .$$

- stable for $g>0$ (repulsive)
- unstable for $g<0$ (attractive).

To see this let's again

Consider Madelung again

As an excersize, show:

- Mass balance eq. same as before

- $$2i\dot{\varphi} = \left(\frac{\partial_t \psi}{\psi} - \frac{\partial_t \psi^*}{\psi^*}\right) =$$
  $$i\frac{\hbar}{2m}\left(\frac{\nabla^2\psi}{\psi} + \frac{\nabla^2\psi^*}{\psi^*} - 2g|\psi|^2\right)$$

contribution of interaction term

$$\Rightarrow \boxed{\begin{aligned} &\partial_t \rho + \nabla\cdot(\rho\bar{u}) = 0, \\ &\partial_t \bar{u} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{\nabla\cdot\overleftrightarrow{P}_Q}{\rho} - \frac{\hbar g}{m}\nabla\rho \end{aligned}}$$

Polytropic pressure $p = \frac{\hbar g}{2m}\rho^{2 \leftarrow \gamma}$

Perturbed uniform state

$\rho = \rho_0 + \tilde{\rho}$, u is small

Linearize; $\tilde{\rho}, \tilde{u} \sim e^{i\vec{k}\cdot\vec{x} - i\omega t}$

$$-i\omega \hat{\rho}_k + \rho_0 i (\vec{k}\cdot\hat{\vec{u}}_k) = 0$$

$$-i\omega \hat{u}_k = -\frac{1}{m} i\vec{k}\hat{Q} - \frac{\hbar g}{m} i\vec{k}\hat{\rho}_k$$

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} , \quad \hat{Q} = -\frac{\hbar^2(-k^2)\cancel{\sqrt{\rho_0}}\frac{\hat{\rho}_k}{2\rho_0}}{2m\cancel{\sqrt{\rho_0}}}$$

$$\omega = \rho_0 \frac{(\vec{k}\cdot\hat{u})}{\hat{\rho}_k} = \rho_0 \frac{1}{\omega}\left(\frac{k^2}{m}\frac{\hat{Q}}{\hat{\rho}_k} + \frac{\hbar g}{m}k^2\right)$$

$$\frac{\omega^2}{\rho_0} = \frac{\hbar^2 k^4}{4m^2\rho_0} + \frac{\hbar g}{m}k^2$$

$$\omega = \sqrt{\frac{\hbar g \rho_0}{m}k^2 + \frac{\hbar^2 k^4}{4m^2}}$$

$g > 0 \rightarrow$ stable

$g < 0 \rightarrow$ unstable for small k.

→ Bogolubov waves

For small k – sound

$$\omega = c_s k, \quad c_s = \sqrt{\frac{\hbar g \rho_0}{m}}.$$

For large k – matter waves

$$\omega = \frac{\hbar}{2m} k^2.$$

→ Instability → collapses.

In classical fluids: $P < 0$ → cavitation

Collapse dominated turbulence – nice project, outside of my course.

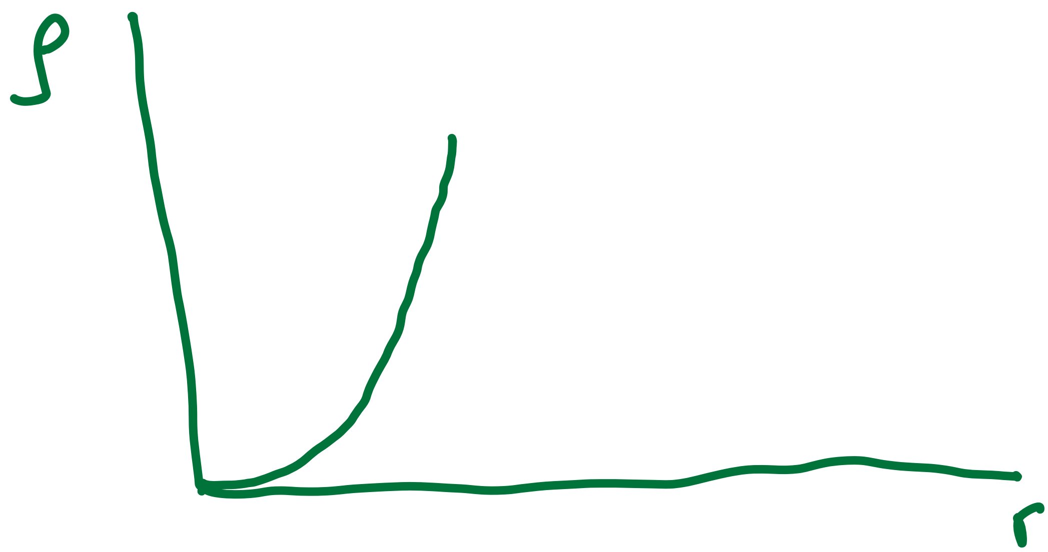
Vortices are important in the stable case

Consider repulsive system $g>0$.

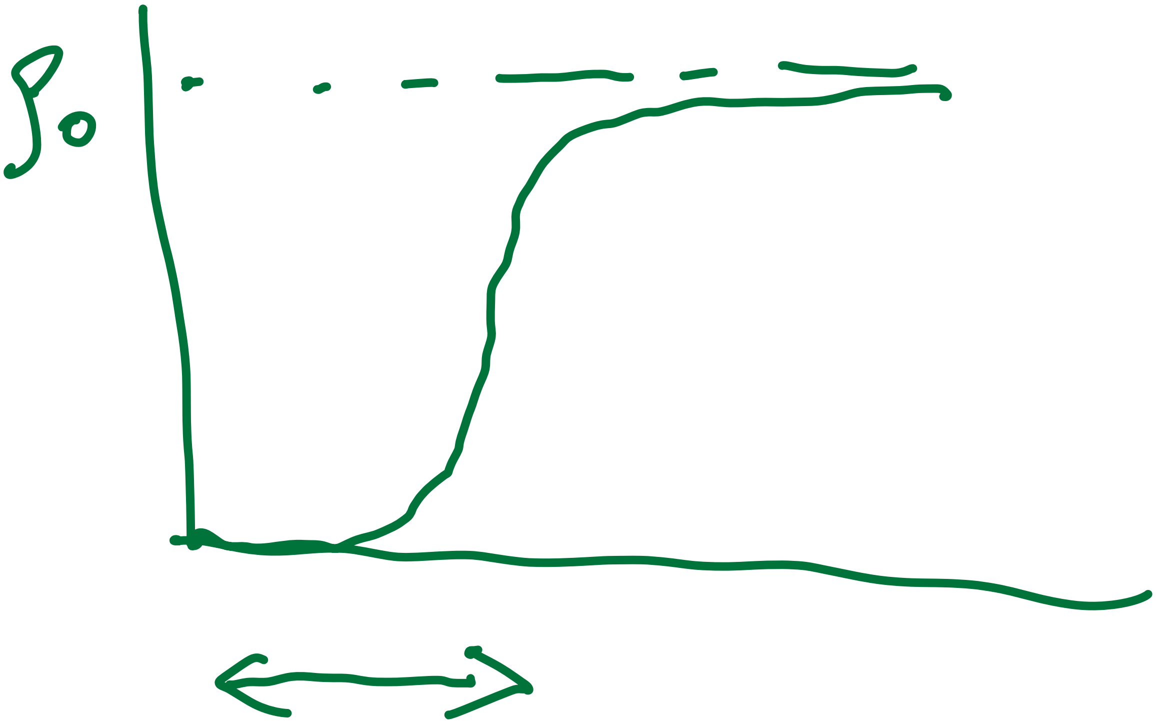
Vortices in GPE

Since the interaction term is small when $|\psi| \to 0$, the GPE vortex has the same structure near its center as th SE vortex:

$$\psi \sim r e^{\pm i\theta}$$



However, as ρ grows, the field feels nonlinearity and ρ "heals" to ρ_0.



healing length $\xi = \frac{\hbar}{\sqrt{g\rho_0}}$

v. core radius

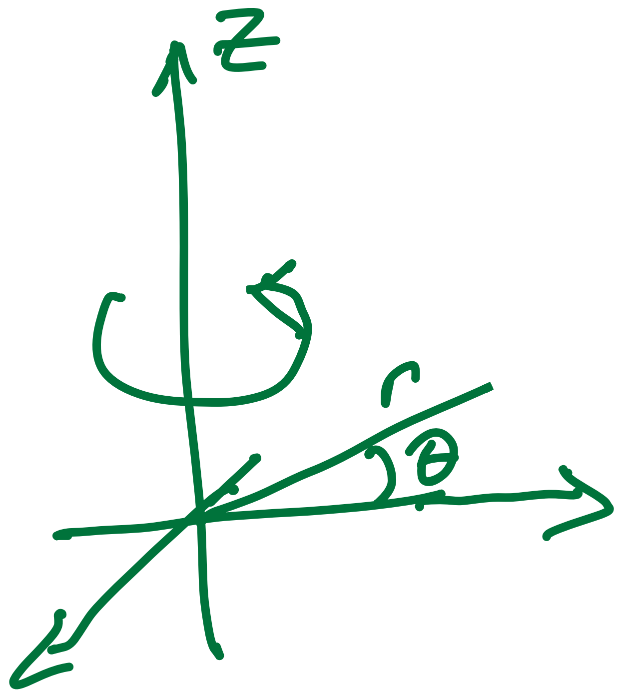
$$i\hbar\,\partial_t \psi = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi$$

Look for solution as

$$\psi(r,\theta) = R(r)\,e^{i\theta - i\frac{g\,\rho_0}{m\hbar}t}$$

$$\nabla^2 \psi = \frac{1}{r}(r\psi_r)_r + \frac{1}{r^2}\psi_{\theta\theta}$$

$$\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2}\right) + g\left(\frac{\rho_0}{m} - R^2\right)R = 0$$



Pitaevskii vortex.

positive circulation.

Negative circulation

$\theta \to -\theta$.

Vortex turbulence

2D

vortex "gas"

ℓ – mean intervortex distance

3D

Vortex tangle

If $\ell \gg \xi$, vortices move as in an ideal classical fluid (Euler eq.)

$u \ll c_s$ incompressible

except: • circulations are quantised and

• v. reconnections, creations and annihilations possible.

Point vortex gas in 2D

Each V. is moved by the velocity field produced by the other vortices.

$\Gamma_j = \pm \frac{h}{m}$

$$\dot{\vec{x}}_k = \sum_{j \neq k}^{N} \frac{\Gamma_j}{2\pi} \frac{\hat{z} \times (\bar{x}_k - \bar{x}_j)}{|\bar{x}_k - \bar{x}_j|^2}$$

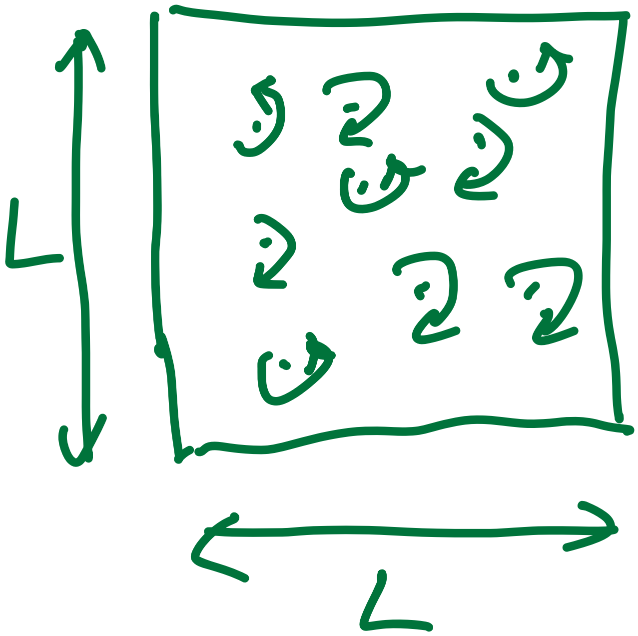
$$H = -\frac{1}{8\pi} \sum_{j \neq k}^{N} \Gamma_j \Gamma_k \ln|\bar{x}_j - \bar{x}_k|$$

$$\dot{x}_k = \frac{\partial H}{\partial y_k} \quad , \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}$$

Skipp, Laurie, SN' 23

Model rigorously derived from GPE.

Onsager '49 Theory



Place N vortices randomly in $L \times L$ square (equal # of $+$ and $-$).

Let $\Phi(E)$ be phase volume with $H_v < E$.

$$H_v = -\frac{1}{4\pi}\sum_{\substack{j,k \\ j\neq k}}^{N} \Gamma_j \Gamma_k \ln \frac{|\bar{x}_j - \bar{x}_k|}{\ell}$$

Obviosly:

$$\Phi(-\infty) = 0, \quad \Phi(+\infty) = L^{2N},$$

$$\Omega(E) = \Phi'(E) > 0.$$

$$\lim_{E \to \pm\infty} \Omega(E) \to 0 \Rightarrow \Omega(E) \text{ is max at some } E_m$$

entropy

$$T = \frac{1}{S'(E)}, \quad S(E) = k_B \ln \Omega(E)$$

$$T = \frac{\Omega}{k_B \Omega'} \rightarrow T<0 \text{ at } E>E_m$$

Striking prediction by Onsager

Canonical distribution

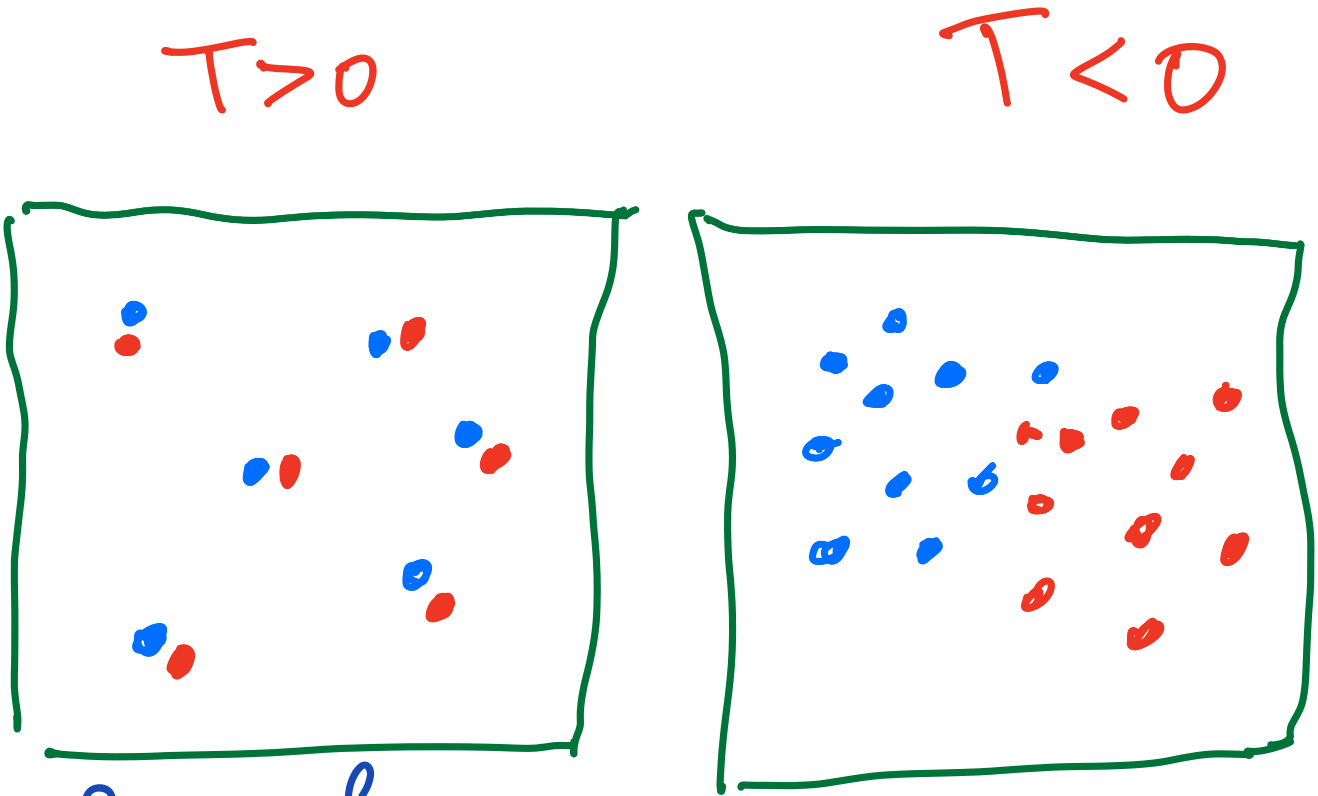
$$P\{\bar{x}_j\} = e^{-H_v\{\bar{x}_j\}/k_B T}$$

To normalize, cutoff at $|\bar{x}_j - \bar{x}_k| < a$

Pairs of like- (opposite-) signed vortices make + (−) contributions to H_v when they are tight $|\bar{x}_j - \bar{x}_k| < \xi$

and opposite when are distant $|\bar{x}_j - \bar{x}_k| > \xi$.

So the most probable realizations:



Gas of dipoles

Large-scale vortex clusters

Vortex annihilation when they meet $|\bar{x}_k - \bar{x}_j| \to \xi$

$\Rightarrow$ Removal of negative E,

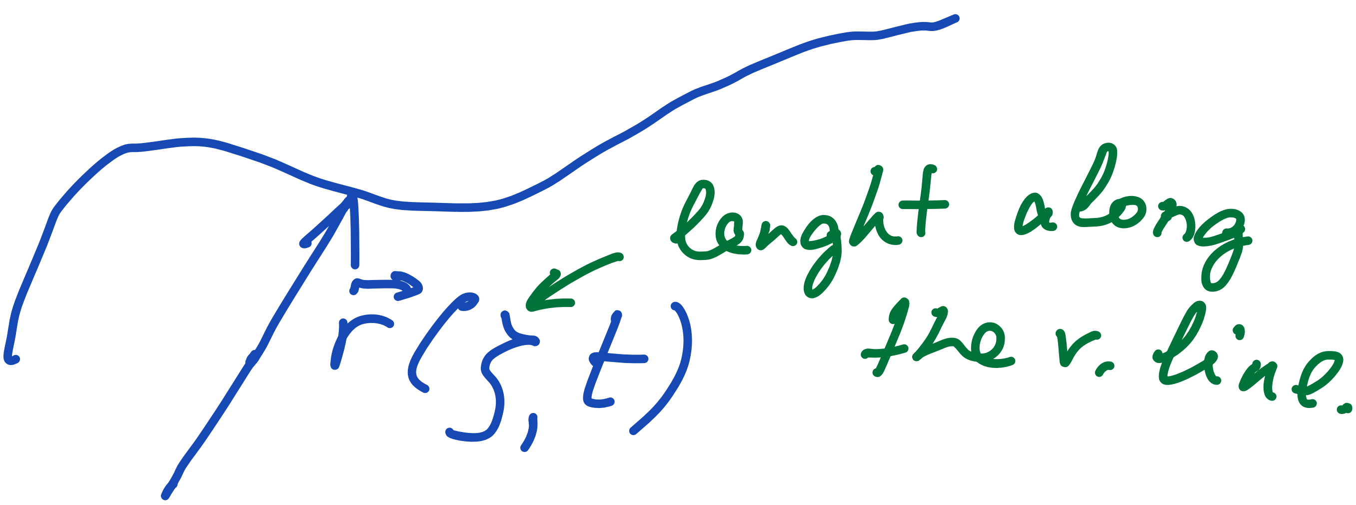
$\Rightarrow$ Growth of E $\Rightarrow$ Clustering

$\Rightarrow$ Vortices in 2D QT must cluster!

3D Vortex tangle

Vortex lines motion described by the Biot-Savart (Da Rios) equation:

$$\frac{\partial \vec{r}}{\partial t} = \frac{\Gamma}{4\pi} \int \frac{(\vec{s} - \vec{r}) \times d\vec{s}}{|\bar{r} - \bar{s}|^3}$$

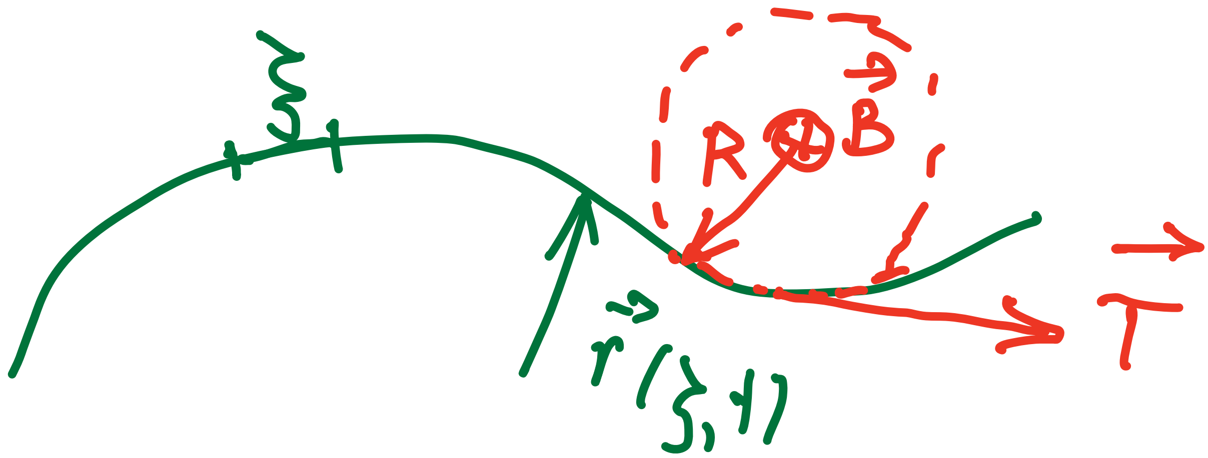


Log singularity at $\bar{s} \to \bar{r}$

Fixed by cutoff at vortex radius:

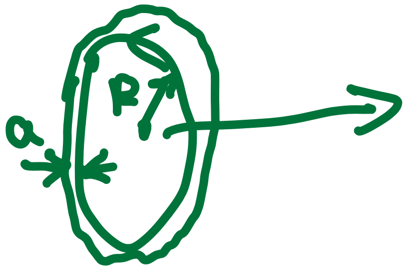
$$|\vec{s} - \vec{r}| < \xi.$$

This BS with cutoff can be rigorously derived from GPE. Bustamante, SN 2015.



Local induction Approximation

LIA



$$V = \frac{\Gamma}{4\pi R} \ln \frac{R}{a}$$

vortex ring

LIA:

- Vortex line moved mostly by nearby line elements in a way similar to vortex ring.

$$\vec{v} \perp \vec{T}, \quad \vec{v} \perp \vec{R}, \quad \vec{v} \parallel \vec{B}.$$

intervortex distance

$$\Lambda = \ln \frac{R}{a} \simeq \ln \frac{\ell}{\xi} \quad \text{const}$$

$$a \simeq \xi$$

$$|\vec{R}| \simeq \frac{1}{|\bar{r}''|}$$

$\partial \xi$

$$\boxed{\dot{\bar{r}} = \frac{\Gamma \Lambda}{4\pi} \, \bar{r}' \times \bar{r}''}$$

Hasimoto Transformation

$$q(\xi,t) = \varkappa(\xi,t)\exp\left(i\int^{\xi}\tau(\lambda,t)\,d\lambda\right)$$

curvature   torsion

$$\boxed{i q_t = q_{\xi\xi} + \frac{1}{2}|q|^2 q}$$

1D Nonlinear Shrödinder

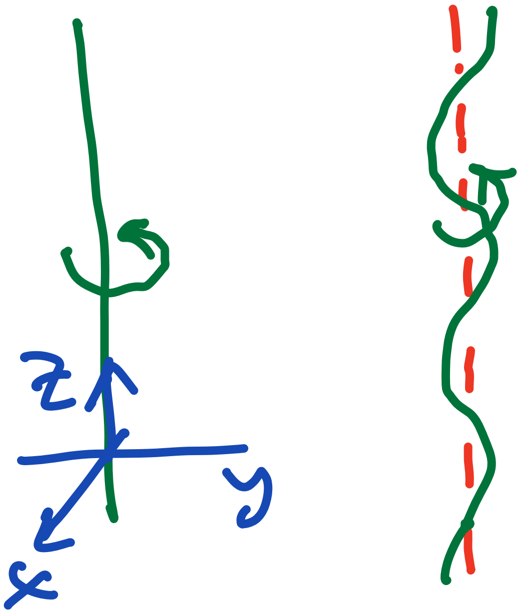
– an integrable model! Solitons etc

Amasing trip NLS ⇒ fluid ⇒ NLS!

Integrability → no cascade

⇒ need to go beyond LIA.

Kelvin Waves



Spiral waves on vortex lines.

$$\vec{r} = (x(z,t),\ y(z,t),\ z)$$

$$z \simeq \xi .$$

$$|\vec{T}| = |\vec{r}\,'| = 1$$

$$a = x + iy$$

$$\dot{a} = \frac{\Gamma\Lambda}{4\pi}\, i\, a''$$

$$a \sim e^{ikz - i\omega t} \rightarrow \boxed{\omega = \frac{\Gamma\Lambda}{4\pi} k^2}$$

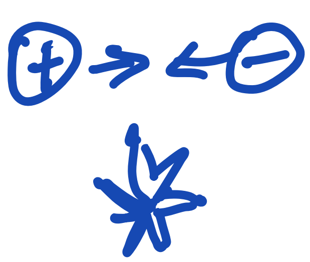
Creation/annihilations + Reconnections in 3D

In classical ideal fluids

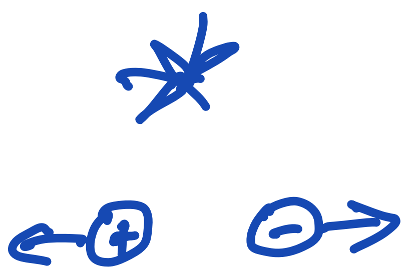
Kelvin circulation theorem:

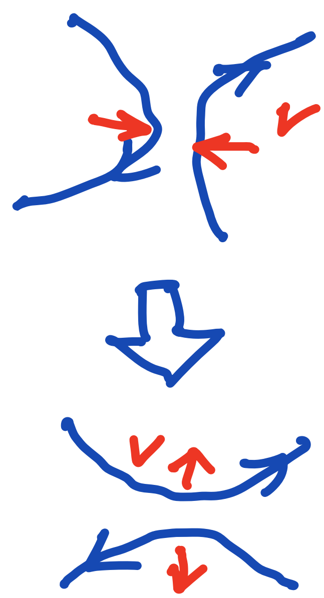
$$\Gamma = \oint_c \vec{u} \cdot d\vec{s} = \text{const}$$

for contours c moving with fluids. This means that v. reconnections and v. creation/annihilation are impossible.



nothing





Quantum pressure term makes these events possible.

Also possible in NSE due to viscosity but in Q. fluids these are non-dissipative events.

Since near vortices ψ is small, reconnections and creations/anihilations can be localy described by the linear SE $(*)$. (Again!)

Consider Ψ:

$$\operatorname{Re}\Psi = ax^2 + by^2 + cz$$

$$\operatorname{Im}\Psi = d(z - vt)$$

$a, b, c, d, v = \text{const}$

SE: $i\partial_t \Psi + \nabla^2 \Psi = 0$

(all coef's rescaled to 1)

or

$$\partial_t \operatorname{Re}\Psi + \nabla^2 \operatorname{Im}\Psi = 0$$

$$-\partial_t \operatorname{Im}\Psi + \nabla^2 \operatorname{Re}\Psi = 0$$

Substitute (*):

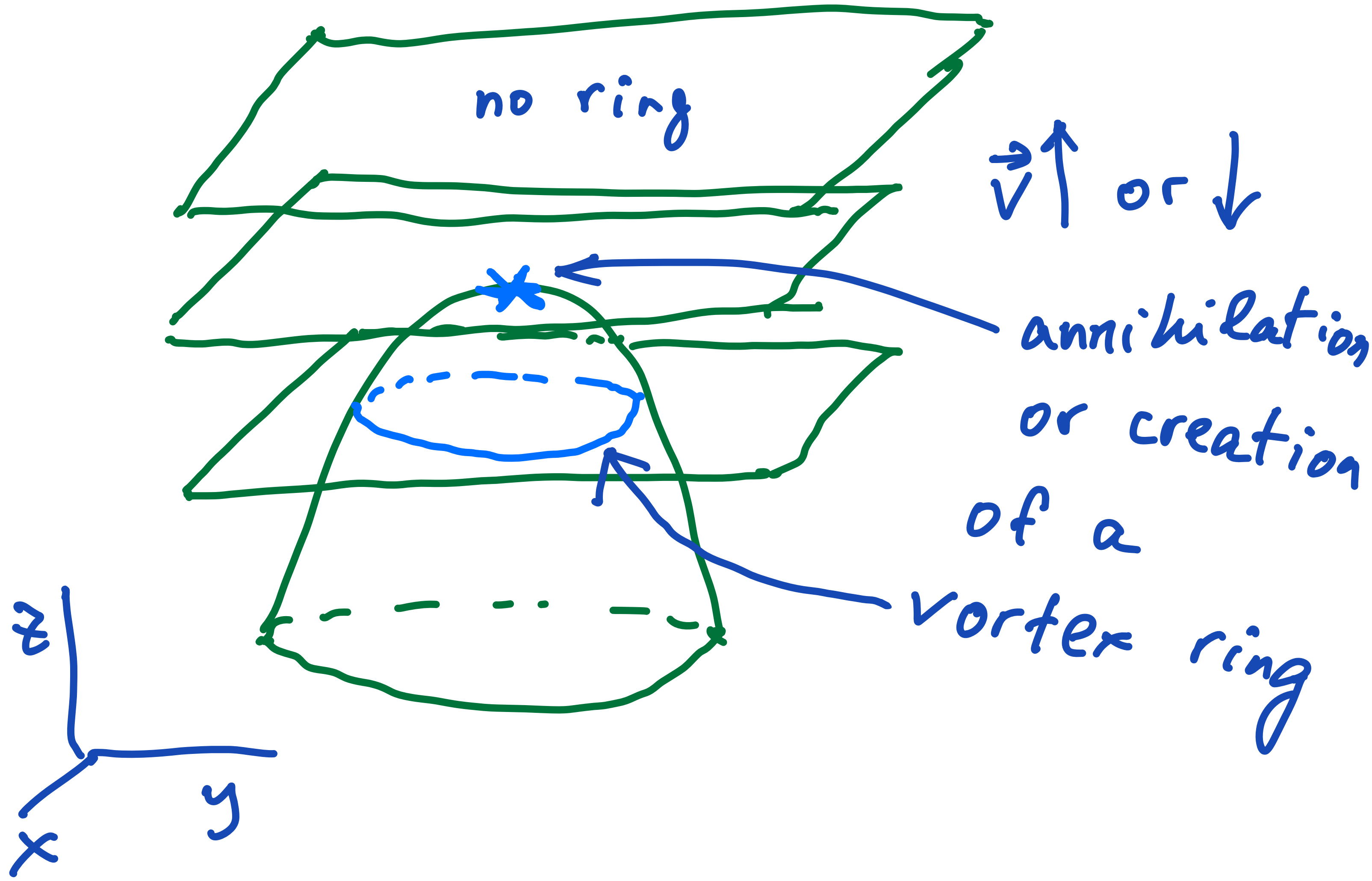
$$\Rightarrow \quad 0 = 0$$

$$+dv + 2(a+b) = 0$$

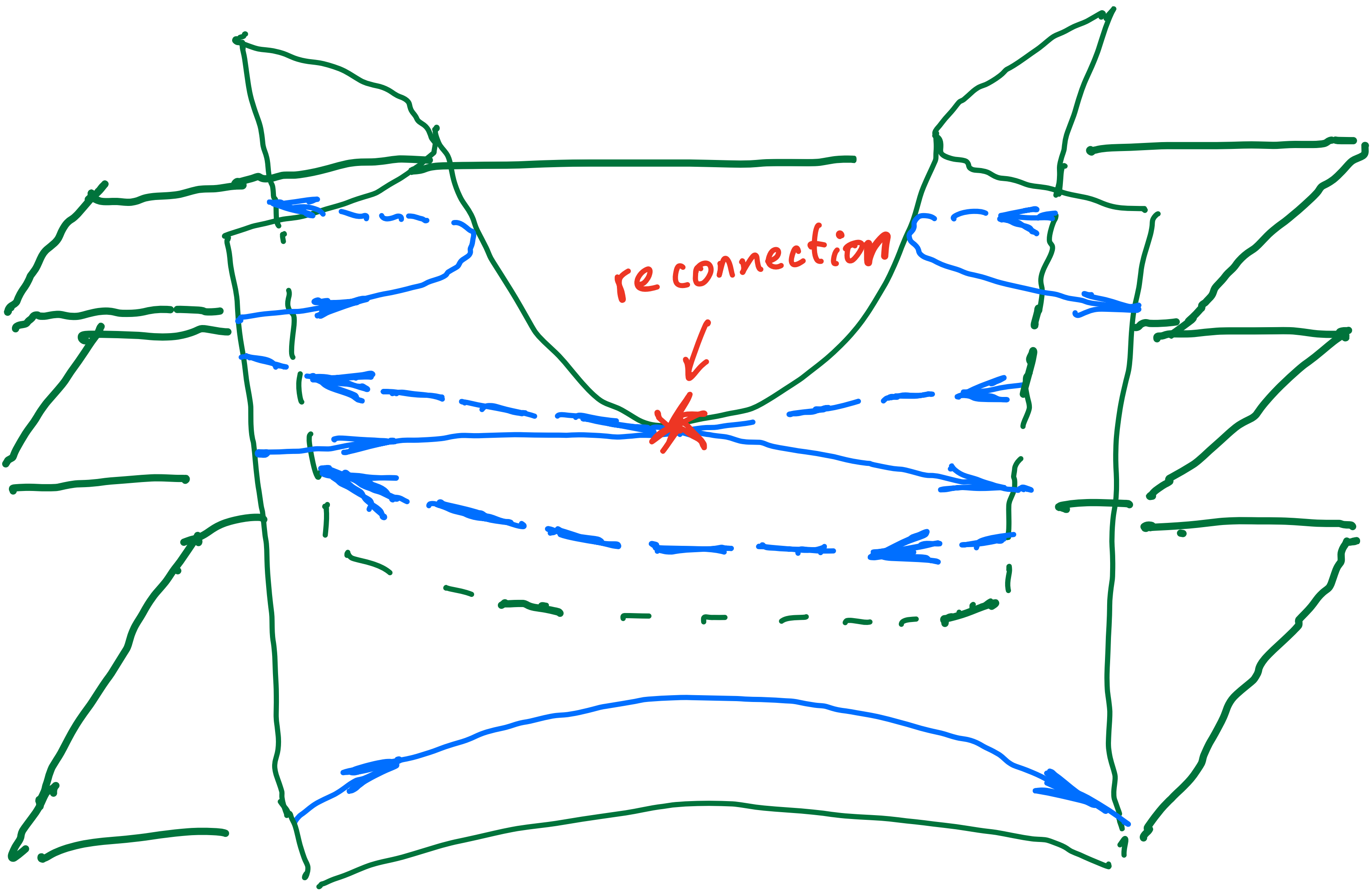
$$\Rightarrow v = -2(a+b)/d$$

Vortices: $\Psi = 0 \rightarrow \mathrm{Re}\Psi = 0 \cap \mathrm{Im}\Psi = 0$

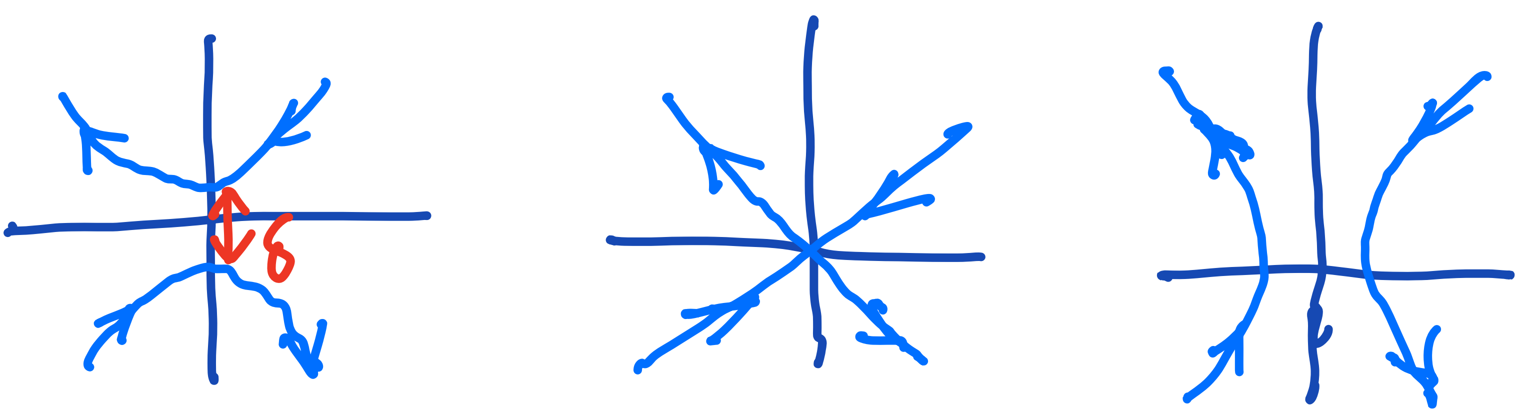
Same sign of a and b:



Different signs of a and b



x-y projection:



According to our solution, the vortex separation near the reconnection point shrinks as $\delta(t) \sim \sqrt{|t_0 - t|}$

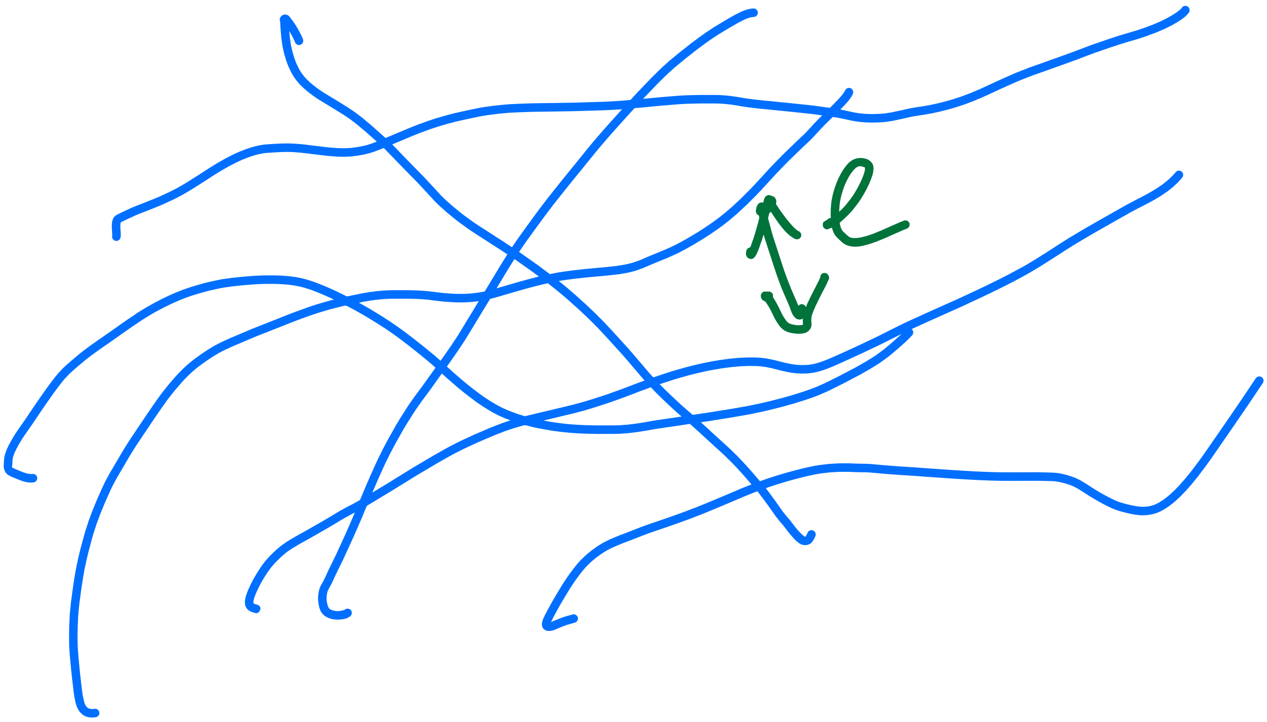
Same for the radius of shrinking/emerging v. ring:

$$R \sim \sqrt{|t_0 - t|}$$

<u>Excersize:</u> find a similar solution describing creation/annihilation of point vortices in 2D:

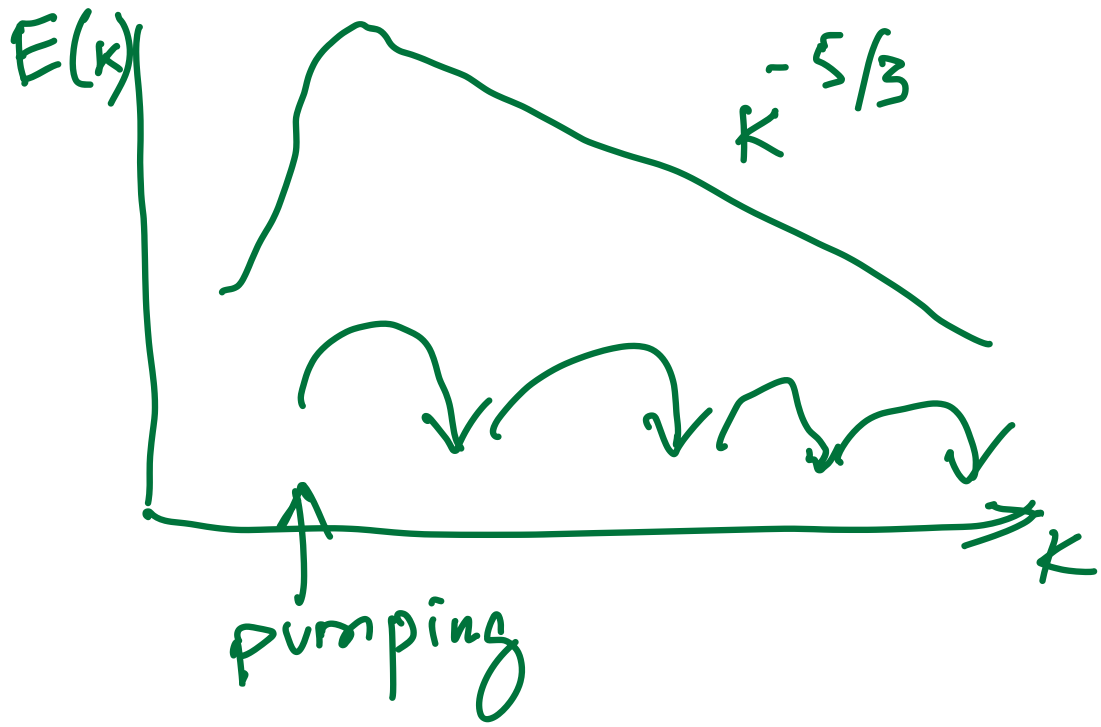
Vortex QT in 3D

Vortex tangle



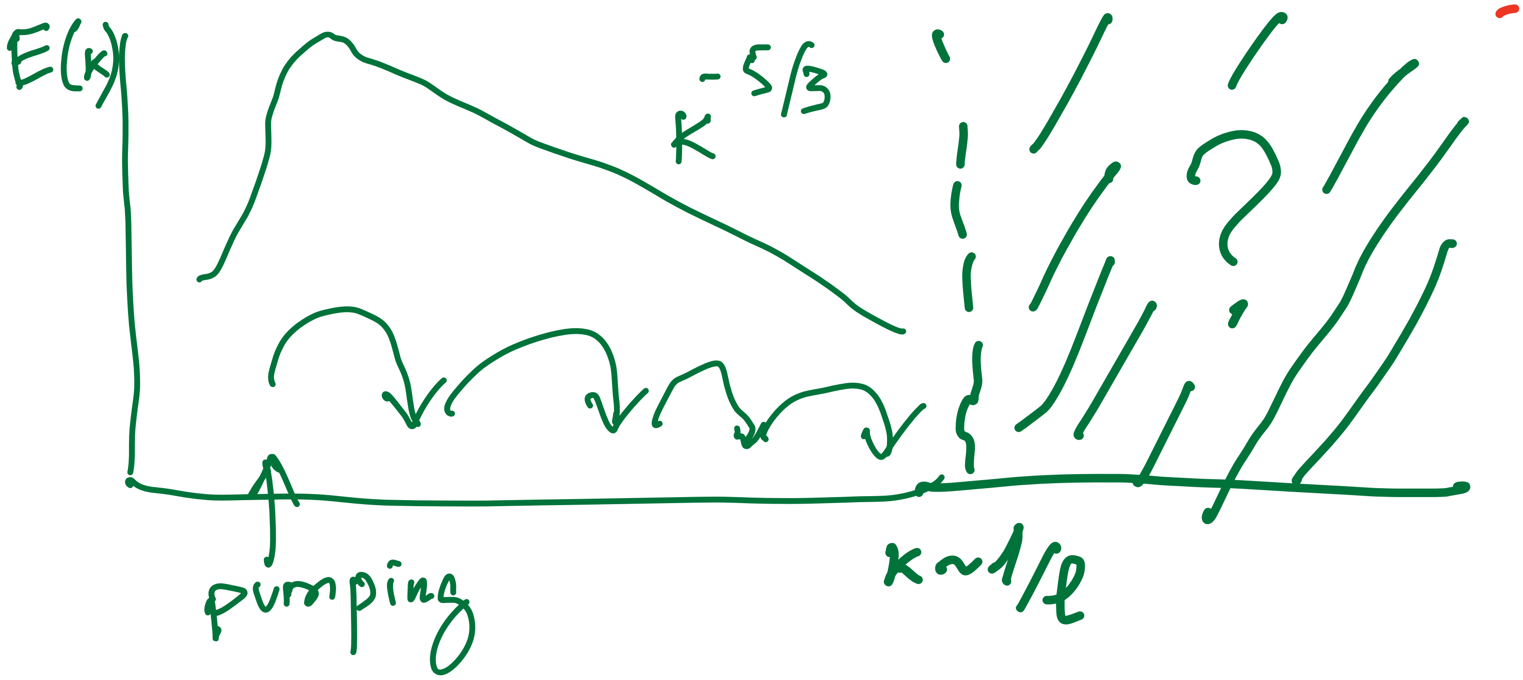
At scales $\gg \ell$, one can coarse grain vorticity. Turbulence becomes similar to classical Kolmogorov K41.

Numerics of 3D GPE and BSF seem to agree with that.

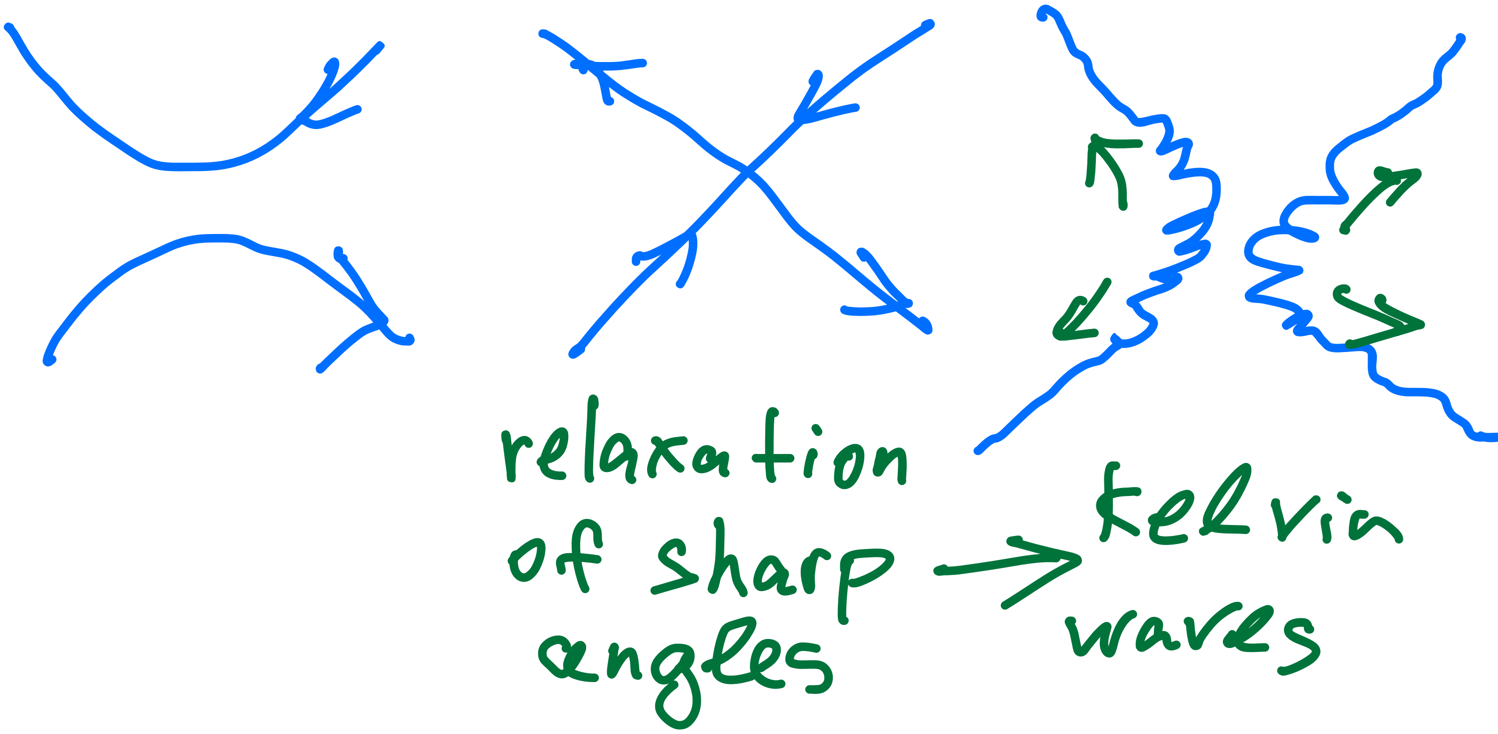


But there is no viscosity so the energy flux inevitably reaches the intervortex scale ℓ.

Then what?



Reconnections are active at $\kappa \sim \ell$



Statistical theory of interacting random KW's — Wave Turbulence theory.

Gentle Wave turbulence

at small nonlinearities resonant interactions dominate

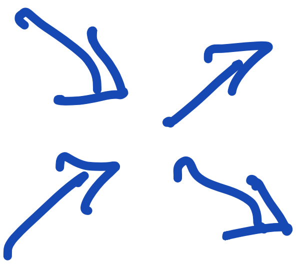
e.g.

$2 \to 1$ 3-wave process

$$\bar{k}_1 + \bar{k}_2 = \bar{k}$$

$$\omega_{k_1} + \omega_{k_2} = \omega_k$$

(Bogolubov waves).



$2 \to 2$ 4-wave process

(Matter waves).

These processes are absent in 1D for $\omega = k^2$

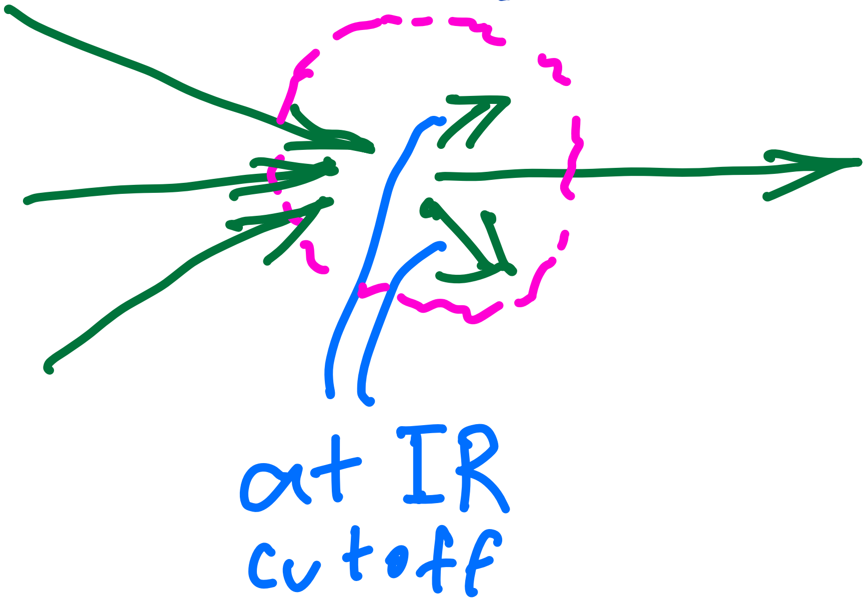
(no solution for resonant conditions)

$\Rightarrow$ KW is a 6-wave process $3 \to 3$ (it has to be even order due to the U(1) symmetry).

It turns out that the locality of interaction hypothesis is broken for the $3 \to 3$ KW turbulence.

Effective theory is 4 wave $3 \to 1$ process.

Effective $3 \to 1$ vertex



For N-wave process

$$\dot{E}_K \sim \underset{\substack{\uparrow \\ E\text{-flux}}}{\epsilon} \sim E_K^{N-1}$$

e.g. 3-wave $\to$

$\dot{E}_K \sim E^2$

Like binary chemical reaction

Another relevant dimensional quantity is Γ : $\dim \Gamma = \frac{\ell^2}{t}$

$\Gamma = \oint u \cdot ds$

$$\dim E_K = \frac{\ell^3}{t^2} \quad , \dim \epsilon = \frac{\ell^2}{t^3}$$

$$\Rightarrow \underset{\ell^5/t^2}{E_K} = C_{KW} \underset{\frac{\ell^{4/3}}{t}}{\epsilon^{1/3}} \underset{\frac{\ell^2}{t}}{\Gamma} \underset{\ell^{+5/3}}{K^{-5/3}}$$

$$5 = \frac{4}{3} + 2 + \frac{5}{3} = \frac{9+6}{3} = \frac{15}{3} \checkmark$$

$$E_k \sim \frac{\ell^{6-d}}{t^2} \sim \frac{\ell^5}{t^2} \quad \varepsilon \sim \frac{\ell^{5-d}}{t^3} \sim \frac{\ell^4}{t^3}$$

$$E = \frac{mu^2}{2}$$

$$\frac{E}{L^d} \sim \frac{\overset{L^3}{m} u^2}{L^d} \sim \frac{\ell^{5-d}}{t^2} \sim \int E_k \, dk$$

$$\Rightarrow E_k \sim \frac{\ell^{6-d}}{t^2}$$

$$\partial_t E_k + \partial_k \varepsilon_k = 0 \rightarrow \dim \varepsilon = \frac{\ell^{5-d}}{t^2}$$

Energy per unit volume

1D energy spectra

$$E = \int E_k \, dk$$

$$\downarrow$$

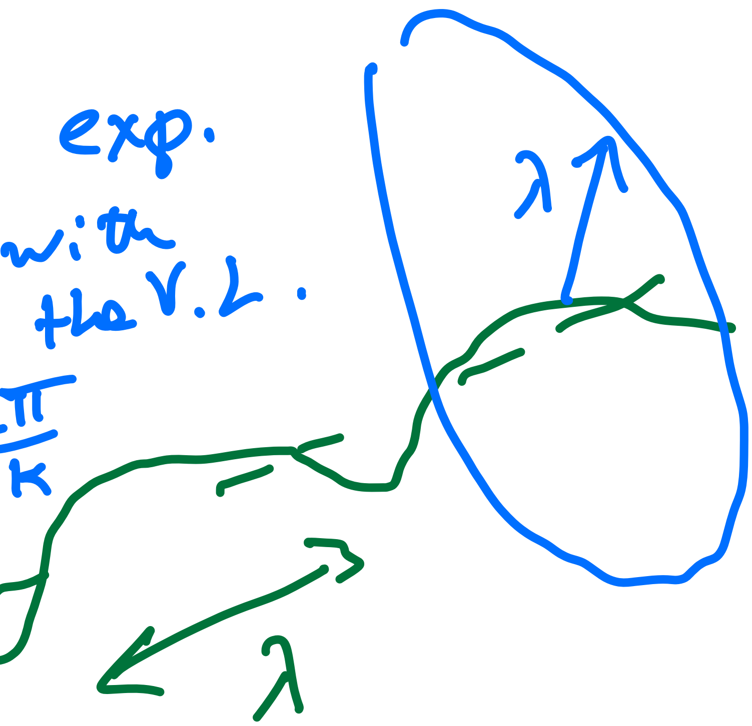
$$\frac{m u^2}{2}$$

Another way to recover dim of E.

mass of fluid involved in motion per unit V.L. length

KW field exp. decaying with distance from the V.L. at $\ell \sim \lambda = \frac{2\pi}{k}$

$\Rightarrow m \sim \rho_0 \lambda^2$

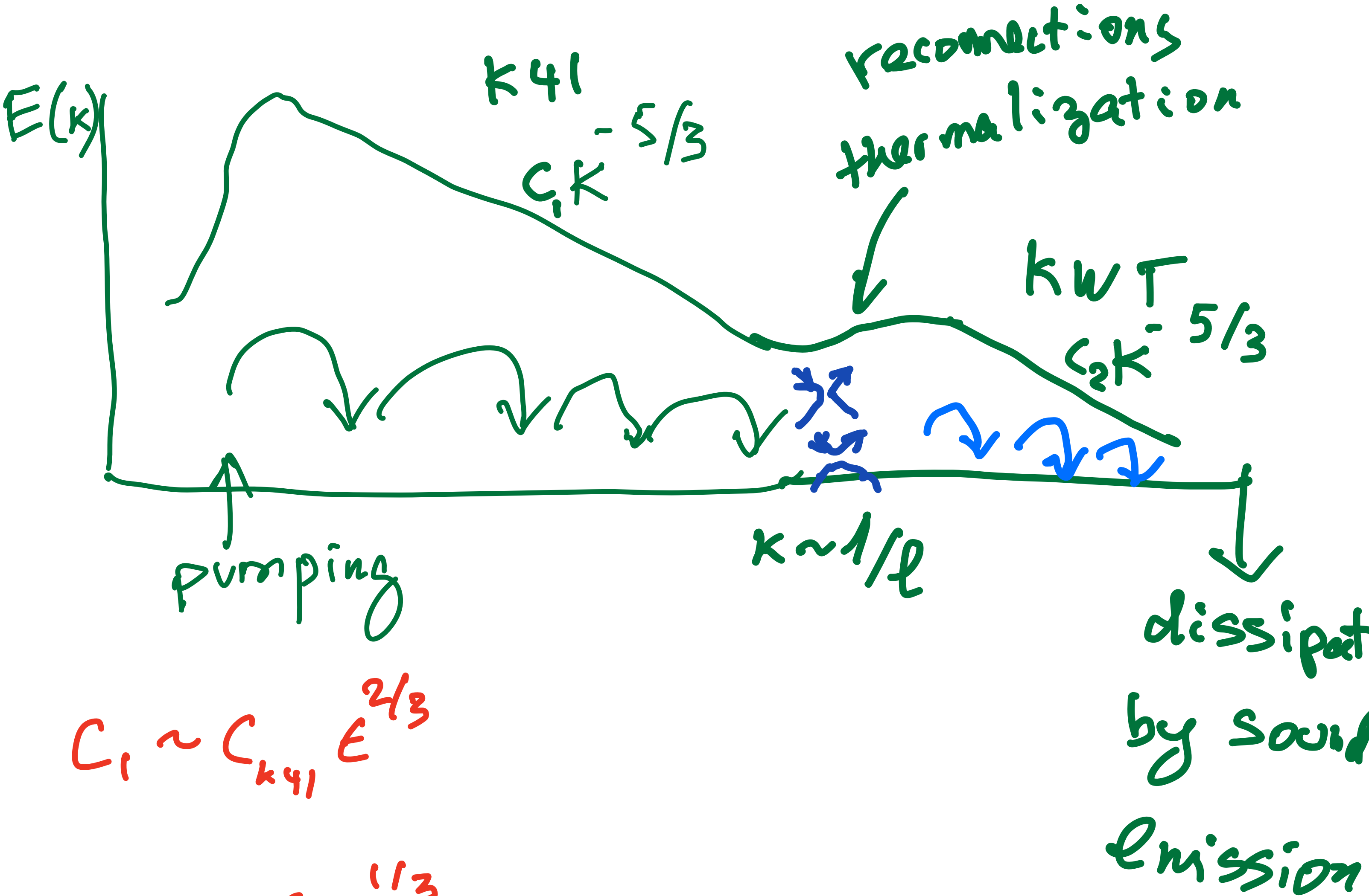


$\Rightarrow \dim E = \rho_0 \frac{L^4}{T^2}$

$\dim E_k = \rho_0 \frac{L^5}{T^2}$.

$\dim \varepsilon_k = \rho_0 \, L^4/T^3$.

Putting together 3D QT:



$C_1 \sim C_{K41}\, \varepsilon^{2/3}$

$C_2 \sim C_{KW}\, \varepsilon^{1/3}$

$C_2 > C_1$ $\leftarrow$ strong turbulence is more efficient to transfer energy than weak KW turbulence.

$\Rightarrow$ Bottleneck near $k\ell \sim 1$: stagnation, partial thermalisation.