Stratified flows

Planetary atmospheres are stratified.
We have gravity

$$\mathbf{f} = -g\rho\,\hat{z}$$

and in a dry atmosphere $(\underline{\nabla}\cdot\underline{v}=0)$

$$\left\{\begin{array}{l} \dfrac{D\underline{v}}{Dt} = -\dfrac{1}{\rho}\underline{\nabla}p - 2\underline{\Omega}\times\underline{v} - g\hat{z} + \nu\nabla^2\underline{v} \\ \dfrac{D\theta}{Dt} = 0 \quad , \quad \dfrac{D\rho}{Dt} = 0 \\ p = \rho R T \longleftarrow \text{(ideal gas)} \end{array}\right.$$

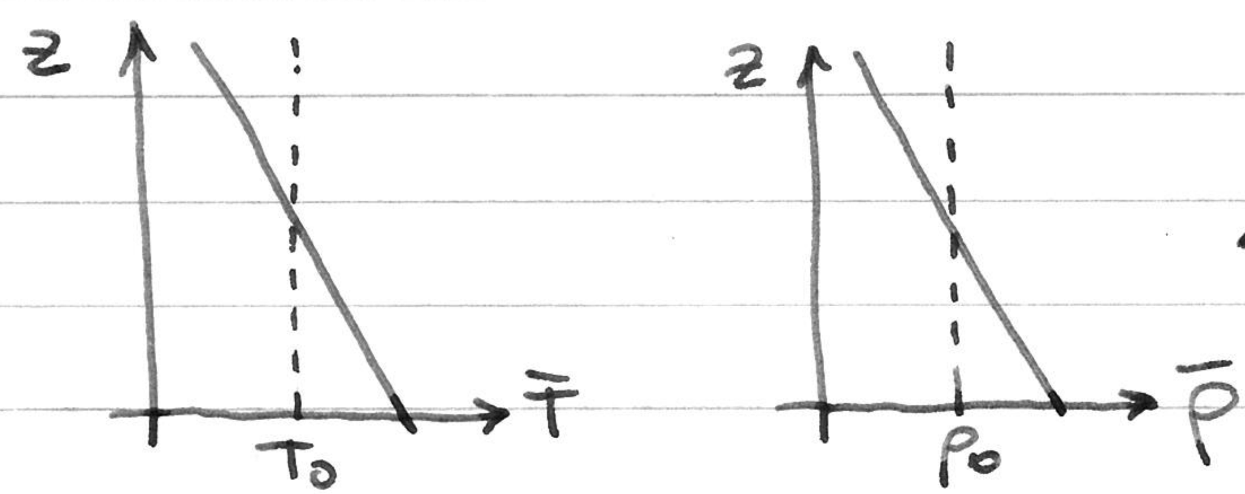
The Boussinesq approximation

Let's set $\underline{\Omega} = 0$ for the moment. We have hydrostatic balance (for the mean pressure)

$$\frac{d\bar{p}}{dz} = -g\bar{\rho}$$

and if the fluid elements are adiabatic

$$\frac{dT}{dz} = -\Gamma_a \qquad \text{with} \quad \Gamma_a = \frac{g}{c_p} \approx 9{,}8\,\frac{K}{m}$$



linear profiles

Let's take

$$\begin{cases} p = \bar{p}(z) + \delta p \\ \rho = \bar{\rho}(z) + \delta \rho \end{cases} \qquad \delta\rho \ll \rho_0$$

then

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = \underbrace{-\frac{1}{\bar{\rho}+\delta\rho}\frac{\partial \bar{p}}{\partial z}\hat{z}}_{=} - \frac{1}{\bar{\rho}+\delta\rho}\underline{\nabla}\delta p - \cancel{g\hat{z}}$$

$\longleftarrow \delta\rho \ll \bar{\rho}$

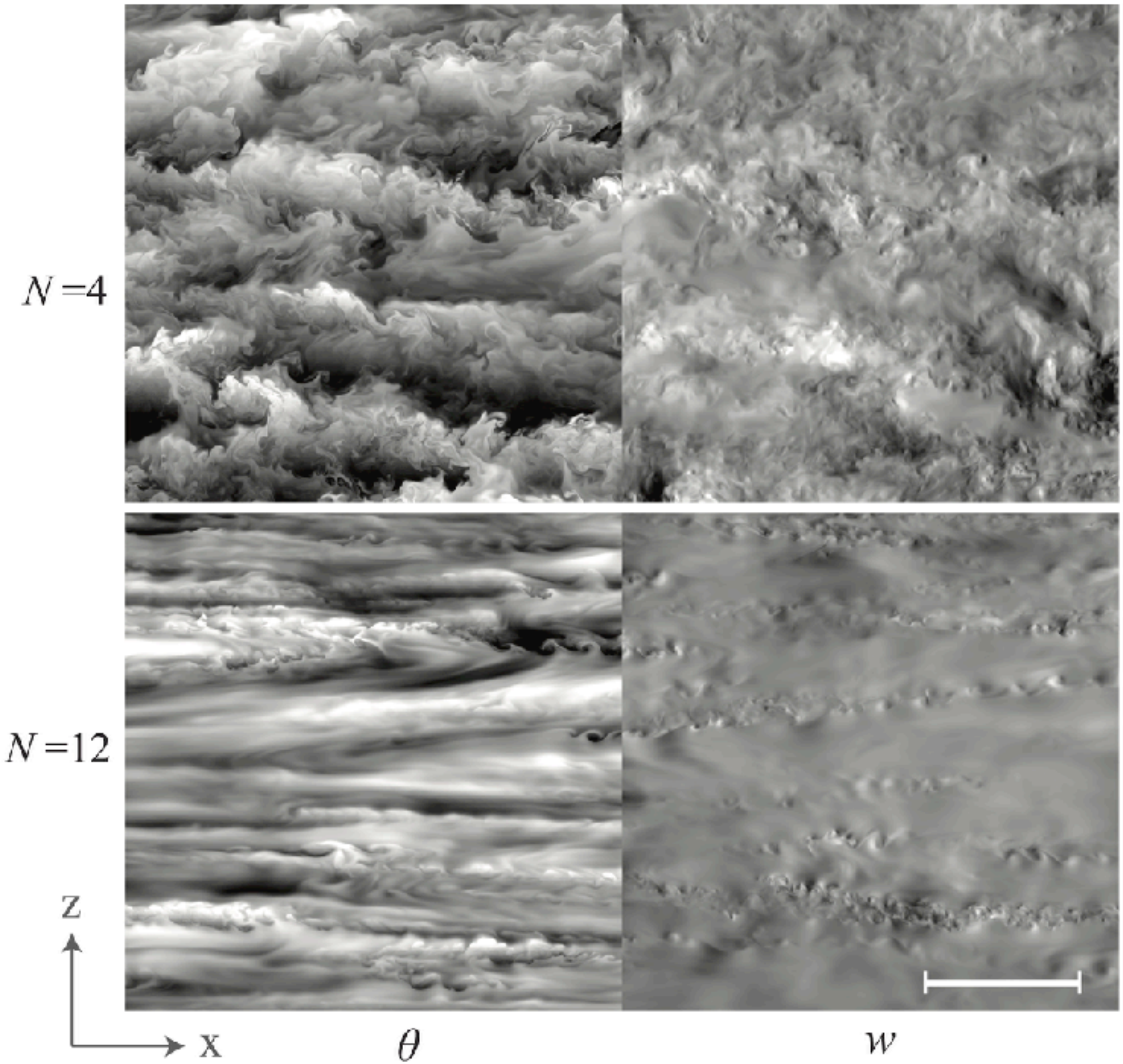
$$\underbrace{-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z}\hat{z}}_{\substack{= \\ \cancel{g\hat{z}}}} + \frac{\delta\rho}{\bar{\rho}^2}\underbrace{\frac{\partial \bar{p}}{\partial z}}_{\substack{= \\ -\bar{\rho}\,\cancel{g}}}$$

In the Boussinesq approximation, we neglect variations in ρ except for the buoyancy

$$-\frac{1}{\bar{\rho}}\delta\rho\, g \approx -\frac{1}{\rho_0}\delta\rho\, g$$

$$-\frac{1}{\bar{\rho}+\delta\rho}\underline{\nabla}\delta p \approx -\frac{1}{\rho_0}\underline{\nabla}\delta p$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\rho_0\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\underline{\nabla}\underline{v}\right) = -\delta\rho\, g\,\hat{\underline{z}} - \underline{\nabla}\delta p \\ &\frac{\partial \delta\rho}{\partial t}+\underline{v}\cdot\underline{\nabla}\delta\rho = -w\frac{\partial\bar{\rho}}{\partial z} \end{aligned}}$$



Rorai et al., PRE (2014)

Internal gravity waves

Let's assume we have a fluid in hydrostatic balance at rest. We apply a small perturbation. Linearizing (and removing the deltas):

$$\begin{cases} \rho_0 \dfrac{\partial \underline{u}}{\partial t} = -\underline{\nabla} p - g\rho \hat{z} & (1) \\[2ex] \dfrac{\partial \rho}{\partial t} = -\dfrac{\partial \bar{\rho}}{\partial z} w \end{cases}$$

Taking the curl of (1)

$$\rho_0 \frac{\partial \underline{\omega}}{\partial t} = -g \underline{\nabla} \times \left(\rho \hat{z} \right) = -g \underline{\nabla} \rho \times \hat{z}$$

Taking $\partial/\partial t$

$$\rho_0 \frac{\partial^2 \underline{\omega}}{\partial t^2} = -g \underline{\nabla}\left(\frac{\partial \rho}{\partial t}\right) \times \hat{z} = g \frac{\partial \bar{\rho}}{\partial z}\left(\underline{\nabla} w \times \hat{z}\right)$$

Taking another curl and using

$$\underline{\nabla} \times \underline{\omega} = -\nabla^2 \underline{v}$$
$$\underline{\nabla} \times \left(\underline{\nabla} w \times \hat{z}\right) = \underline{\nabla}\left(\frac{\partial w}{\partial z}\right) - \nabla^2 w \hat{z}$$

$$\Rightarrow \boxed{\rho_0 \frac{\partial^2}{\partial t^2} \nabla^2 w = g \frac{\partial \bar{\rho}}{\partial z}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) w}$$

This looks again like a wave equation!

stationary solutions $\left(\frac{\partial}{\partial t} = 0\right)$ have $\nabla^2 - \frac{\partial^2}{\partial z^2} = \nabla_\perp^2 = 0$

Solutions are independent of x and y (<u>blocking</u>!).

Taking $\quad w = w_0 \, e^{i(\underline{k} \cdot \underline{x} - \sigma t)}$

$$\Rightarrow \quad \sigma^2 k^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}\underbrace{\left(k^2 - k_z^2\right)}_{k_x^2 + k_y^2 = k_\perp^2}$$

$$\Rightarrow \quad \sigma^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}\frac{k_\perp^2}{k^2}$$

Defining

$$\boxed{N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}}}$$

Brunt-Väisälä frequency

$$\Rightarrow \quad \boxed{\sigma^2 = N^2 \frac{k_\perp^2}{k^2}}$$

Dispersion relation

or $$\sigma = \pm N \frac{k_\perp}{k}$$

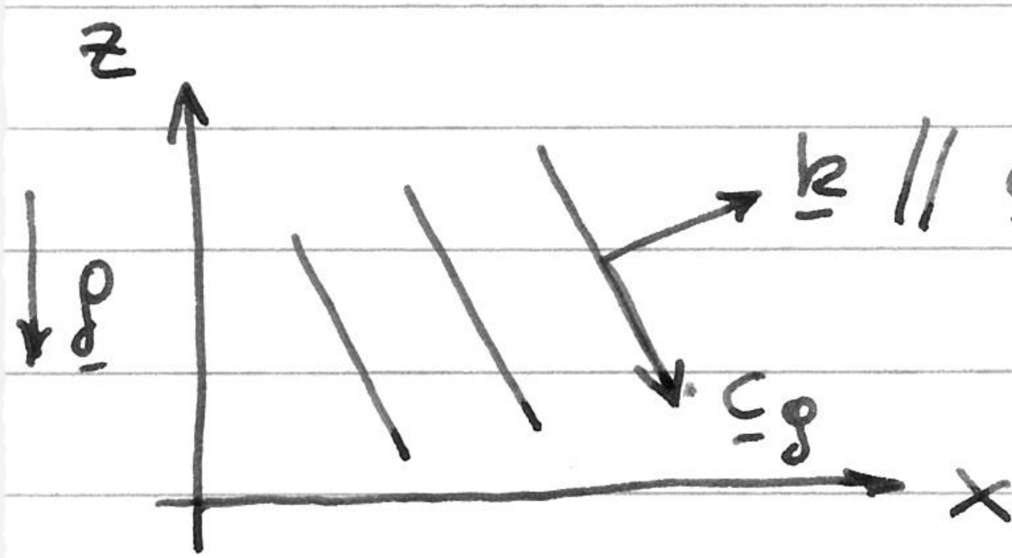
If N real $\left(\frac{\partial \bar{\rho}}{\partial z} < 0\right)$ we have waves

But if $\frac{\partial \bar{\rho}}{\partial z} > 0 \Longrightarrow$ instability (convection)

From the dispersion relation

$$\underline{c}_\varphi = \frac{\sigma \underline{k}}{k^2} = \pm \frac{N k_\perp}{k^3} \underline{k}$$

$$\underline{c}_g = \underline{\nabla}_k \sigma = \pm \frac{N}{k^3 k_\perp} \left[\underline{k} \times \left(\underline{k} \times k_z \hat{z} \right) \right]$$



Also

$$\rho' = \mp i \frac{\rho_0 N}{g} \frac{k}{k_\perp} w_0$$

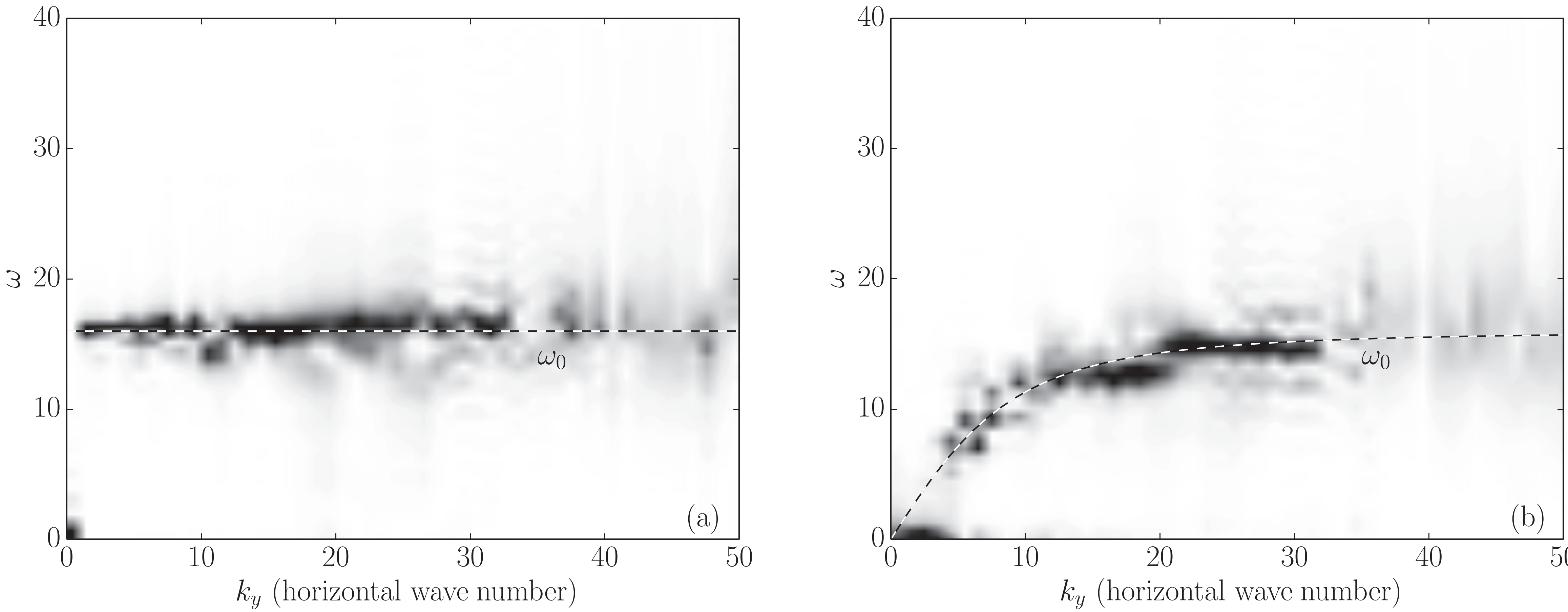
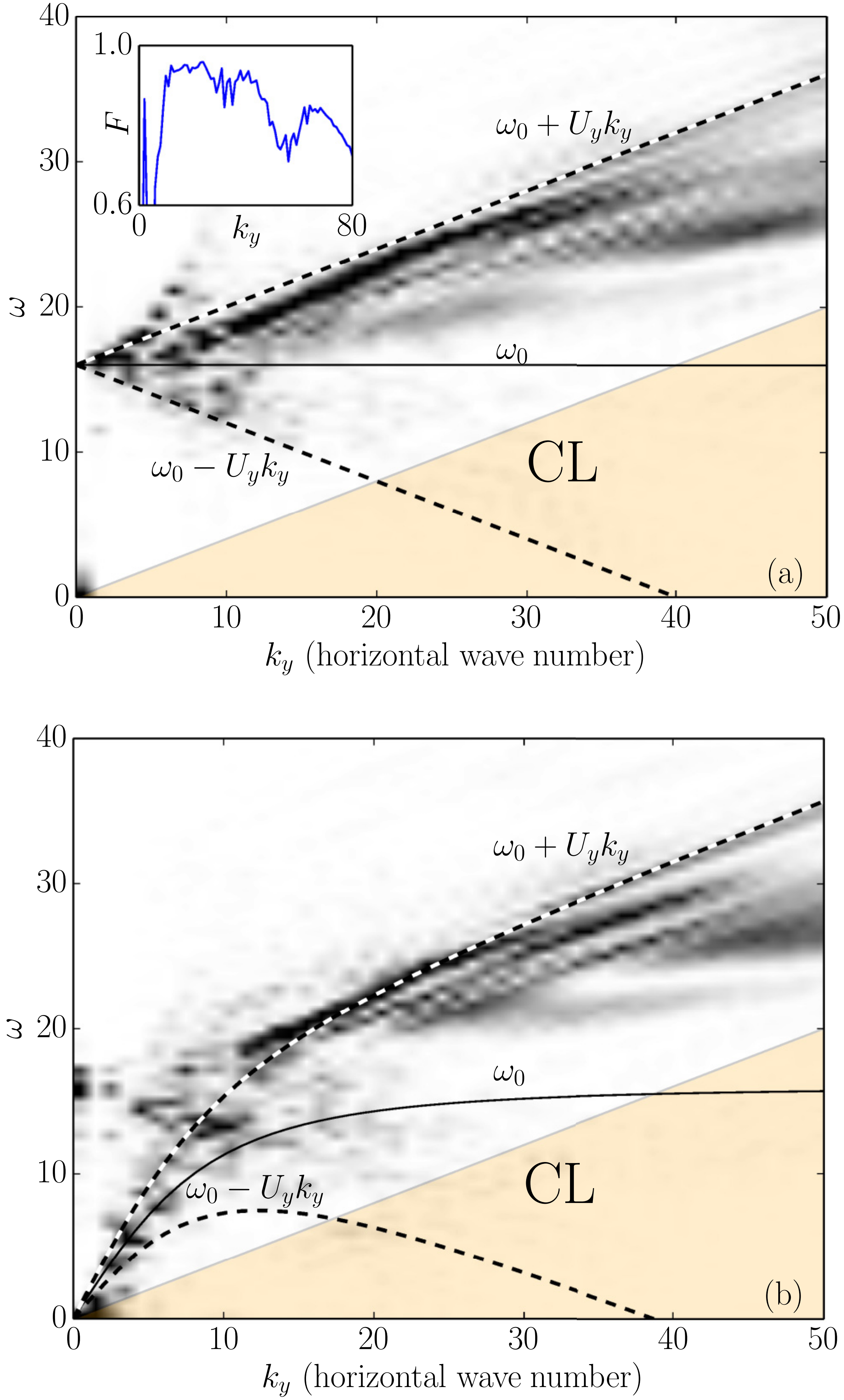
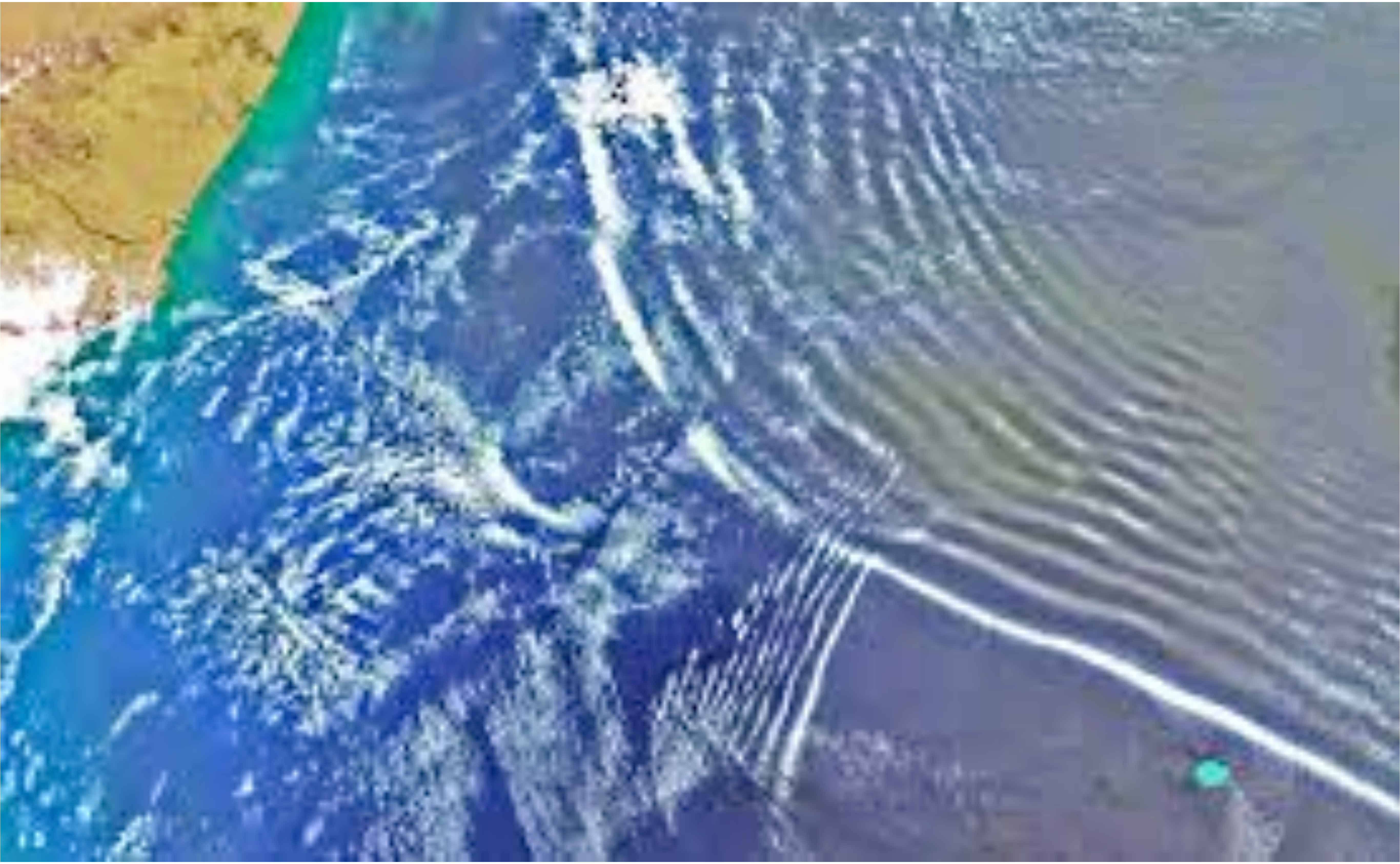


FIG. 8. Space- and time-resolved spectrum of the potential energy $E_\theta(k_x = 0, k_y, k_z, \omega)$ [normalized by $E_\theta(\mathbf{k})$] for two values of k_z: (a) $k_z = 0$ and (b) $k_z = 10$. There is no mechanical forcing in this simulation but a randomly generated, isotropic, and constant-in-time external source of temperature fluctuations. The flow is then dominated by gravity waves, with an almost negligible large-scale horizontal flow. Note the absence of CL absorption and the negligible Doppler shift; most of the energy is concentrated along the dispersion relation for the waves.

Clark Di Leoni & Mininni, PRE (2015)



Clark Di Leoni & Mininni, PRE (2015)



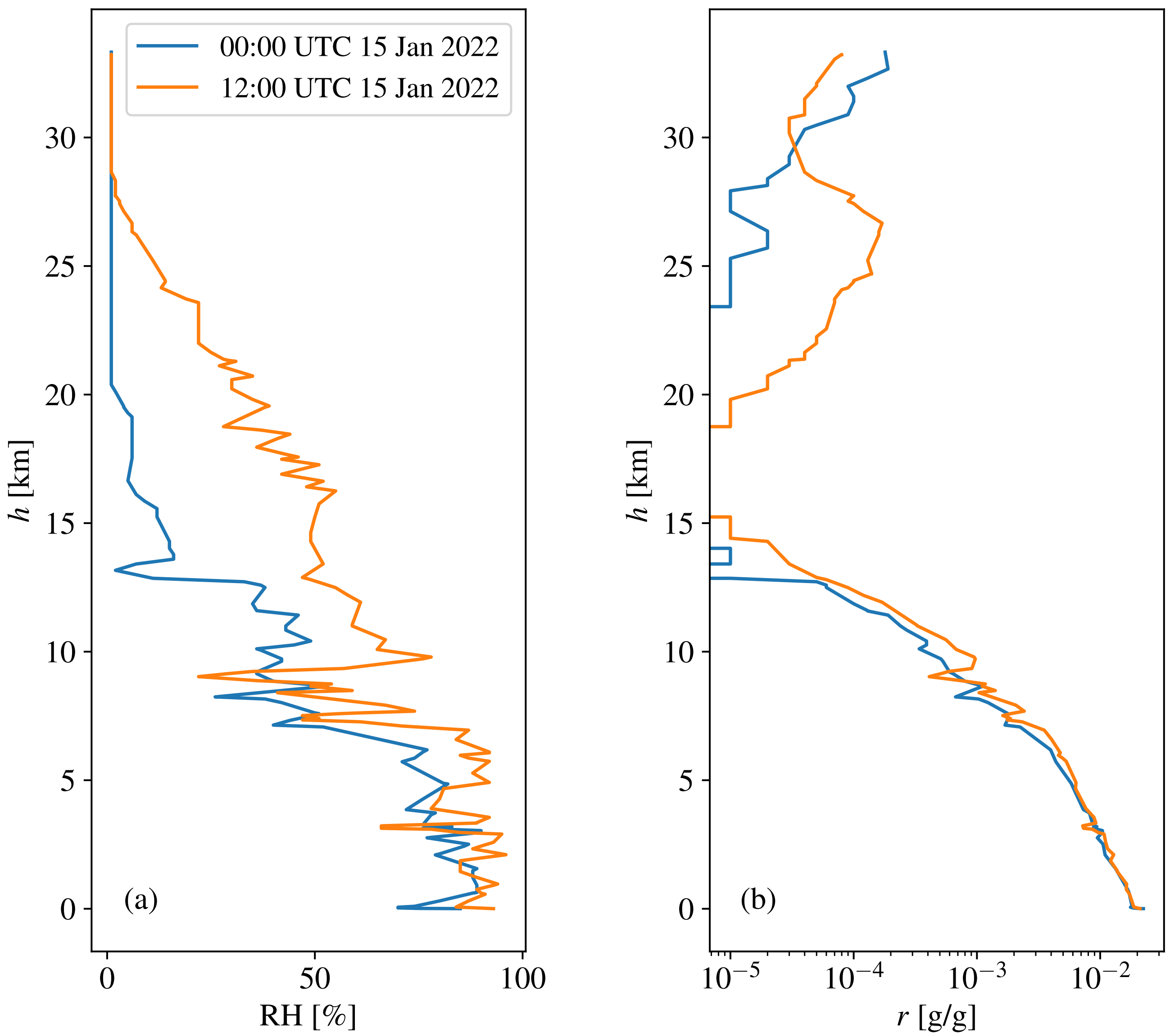
Froude number

Taking $\xi = \sqrt{-\frac{g}{\rho_0}\left(\frac{\partial \bar{\rho}}{\partial z}\right)^{-1}}\,\rho$

$$\Rightarrow \begin{cases} \dfrac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\dfrac{1}{\rho_0}\underline{\nabla} p - N\xi\hat{z} + \nu\nabla^2\underline{v} \\ \dfrac{\partial \xi}{\partial t} + \underline{v}\cdot\nabla\xi = Nw + K\nabla^2\theta \end{cases}$$

Taking $\dfrac{|\underline{v}\cdot\nabla\underline{v}|}{|N\xi|} \sim \dfrac{U^2}{L}\dfrac{1}{NU} \sim \boxed{\dfrac{U}{LN} = Fr}$

The effect of moisture is often neglected in turbulence studies:

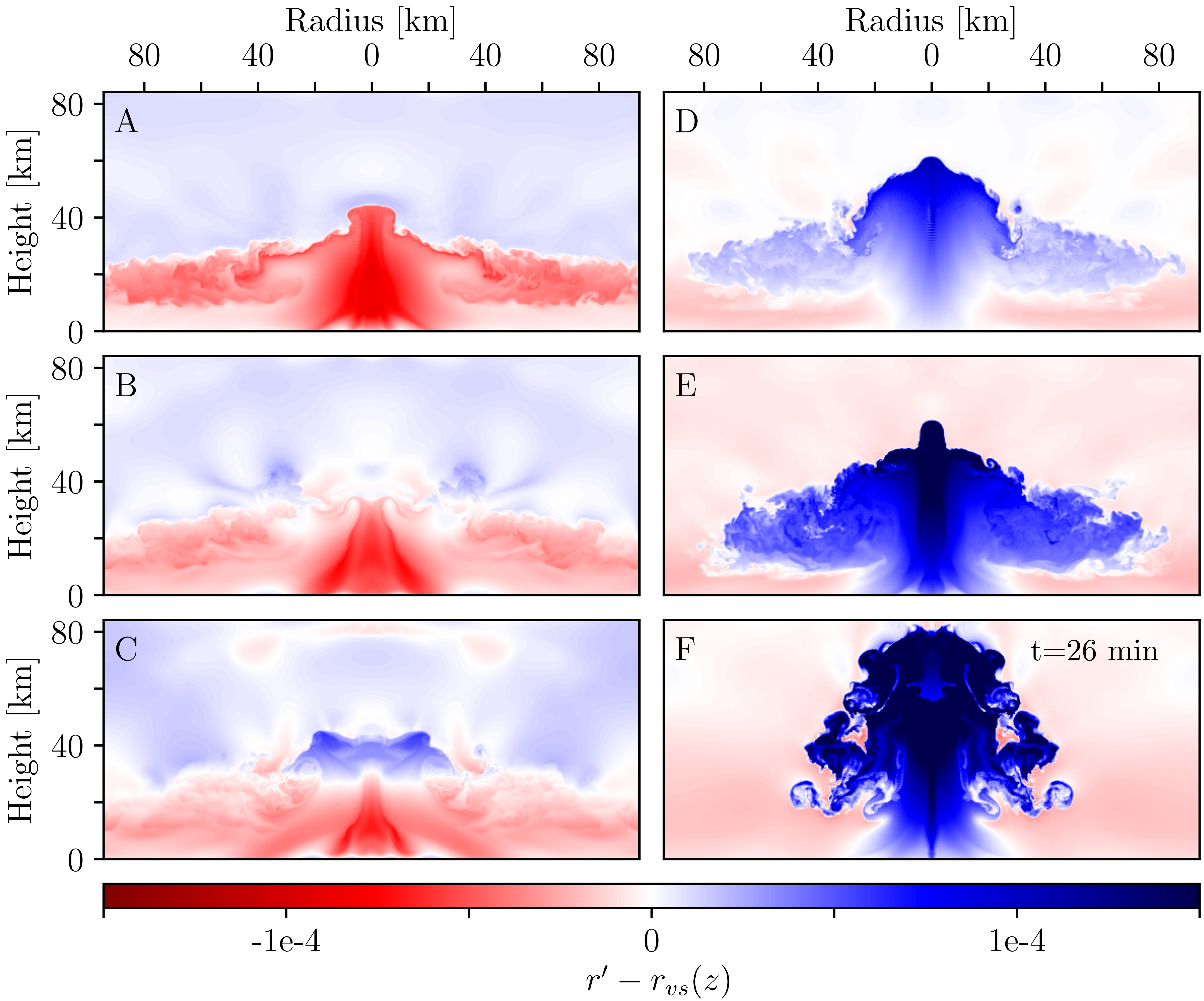


Marsico, Smith & Stechmann (2019):

We now need an unsaturated and a saturated phase, plus a parameterization for the phase transition:

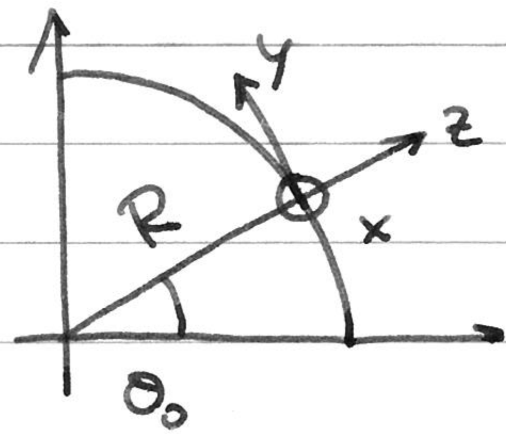
$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla(p/\rho_0) + (b_u\Theta_u + b_s\Theta_s)\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u},\\
\frac{\partial b_u}{\partial t} + \mathbf{u}\cdot\nabla b_u &= -N_u^2 w + \kappa\nabla^2 b_u + s_u,\\
\frac{\partial b_s}{\partial t} + \mathbf{u}\cdot\nabla b_s &= -N_s^2 w + \kappa\nabla^2 b_s + s_s,
\end{aligned}
$$

$$
\begin{aligned}
b_u &= g\left[\frac{\theta_e'}{\theta_0} + \left(R^* - \frac{L}{c_p\theta_0}\right) r'\right],\\
b_s &= g\left[\frac{\theta_e'}{\theta_0} + \left(R^* - \frac{L}{c_p\theta_0} + 1\right) r_{vs}(z) - r'\right].
\end{aligned}
$$



Radius [km]
80
40
0
40
80
Radius [km]
80
40
0
40
80
Height [km]
80
40
0
A
B
C
D
E
F
t=26 min
-1e-4
0
1e-4
$r' - r_{vs}(z)$

Quasi-geostrophic approximation



$$f \simeq 2\Omega \left[s\theta_0 + c\theta_0 (\theta - \theta_0) + \dots \right]$$

$$\approx f_0 + \beta y$$

where $f_0 = 2\Omega s\theta_0$

$$\beta = \frac{2\Omega}{R} c\theta_0$$

From Boussinesq, to lowest order $(\beta = 0)$

$$R_0 \ll 1) \quad \cancel{\underbrace{\partial_t \underline{u} + \underline{u} \cdot \underline{\nabla}_\perp \underline{u}}^{U^2/L}} + w\,\partial_z \underline{u} = -f_0 \hat{z} \times \underline{u} - \frac{1}{\rho_0} \underline{\nabla}_\perp p$$

$$\delta \ll 1) \quad \cancel{\partial_t w + \underline{u} \cdot \underline{\nabla}_\perp w + w\,\partial_z w} = -\frac{1}{\rho_0} \partial_z p - N\zeta$$

$w \sim \delta U$

hydrostatic balance

geostrophic balance

from geostrophic balance:

$$-f_0 u = \frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad f_0 v = \frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

Taking $\boxed{\psi = \frac{p}{f_0 \rho_0}}$

$$\Longrightarrow \quad \underline{v} = \underline{u}_g = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0\right) = -\underline{\nabla} \times (\psi \hat{z})$$

From $\underline{\nabla} \cdot \underline{v} = 0$ and boundary conditions $w = 0$

And from hydrostatic balance $\boxed{\zeta_g = -\frac{f_0}{N}\frac{\partial \psi}{\partial z}}$

Let's go to the next order in an expansion in some small parameter:

$$x = L x' \qquad \nabla_\perp = \frac{\nabla_\perp'}{L} \qquad \zeta = U \xi'$$

$$z = H z' \qquad \underline{v} = U \underline{v}'$$

$$t = \frac{L}{U} t' \qquad \partial_z = \frac{\partial_z'}{H} \qquad w = \delta U w'$$

$$f_0 = \frac{U}{L R_0} f_0' \qquad \beta = \frac{U}{L^2} \beta'$$

$R_0 = \frac{U}{Lf}$ $\qquad$ $\beta y \sim \frac{U}{L}$

To lowest order pressure is balanced by Coriolis in $\perp$:

$$\frac{1}{\rho_0}\nabla_\perp p \sim \frac{1}{L}\frac{P}{\rho_0} \sim \frac{U^2}{L R_0} \Longrightarrow \frac{p}{\rho_0} = \frac{U^2}{R_0}\frac{p'}{\rho_0'}$$

And in z:

$$\frac{1}{\rho_0}\partial_z p \sim \frac{1}{H}\frac{P}{\rho_0} \sim N\xi \Longrightarrow N = \frac{U}{R_0 H} N'$$

$$N \sim \frac{U^2}{R_0}\frac{1}{UH} \sim \frac{U}{R_0 H}$$

In Boussinesq:

$$\frac{\cancel{U^2}}{\cancel{L}} R_0 \left(\partial_{t'} \underline{u}' + \underline{u}' \cdot \underline{\nabla}'_\perp \underline{u}' + w' \partial'_z \underline{u}' \right) = - \frac{\cancel{U^2}}{\cancel{L}} f_0' \hat{z} \times \underline{u}' - \frac{\cancel{U^2}}{\cancel{L}} R_0 \beta' y' \hat{z} \times \underline{u}'$$

$$- \frac{\cancel{U^2}}{\cancel{L}} \frac{1}{\rho_0'} \underline{\nabla}'_\perp p'$$

$$\frac{\cancel{U^2}}{\cancel{L}} \left(\partial_{t'} \zeta' + \underline{u}' \cdot \underline{\nabla}'_\perp \zeta' + w' \partial'_z \zeta' \right) = \frac{U^2 \cancel{\delta}}{R_0 \cancel{H}} N' w'$$

We now expand

$$\underline{u}' = \underline{u}_g' + R_o \underline{u}_1' + R_o^2 \underline{u}_2' + \dots \qquad \underline{v} = (\underline{u}, w)$$

$$w' = \cancel{w_g} + R_o w_1 + \dots$$

$$\xi' = \xi_g + R_o \xi_1 + \dots$$

To first order

$$\boxed{D_g' \underline{u}_g' = -f_0' \hat{z} \times \underline{u}_1' - \beta' y' \hat{z} \times \underline{u}_g'} \qquad D_g' = \frac{\partial}{\partial t'} + \underline{u}_g' \cdot \underline{\nabla}_\perp'$$

From $\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \underline{\nabla}_\perp' \cdot \underline{u}_1' + \frac{\partial w'}{\partial z'} = 0$

and $\boxed{D_g' \xi_g' = N' w_1}$

We can recover eqs. with units by removing the primes. Taking the curl of the first eq. and using

$$\underline{\nabla} \times \underline{u}_g = \nabla_\perp^2 \psi \, \hat{z}$$

The eq. can be written as

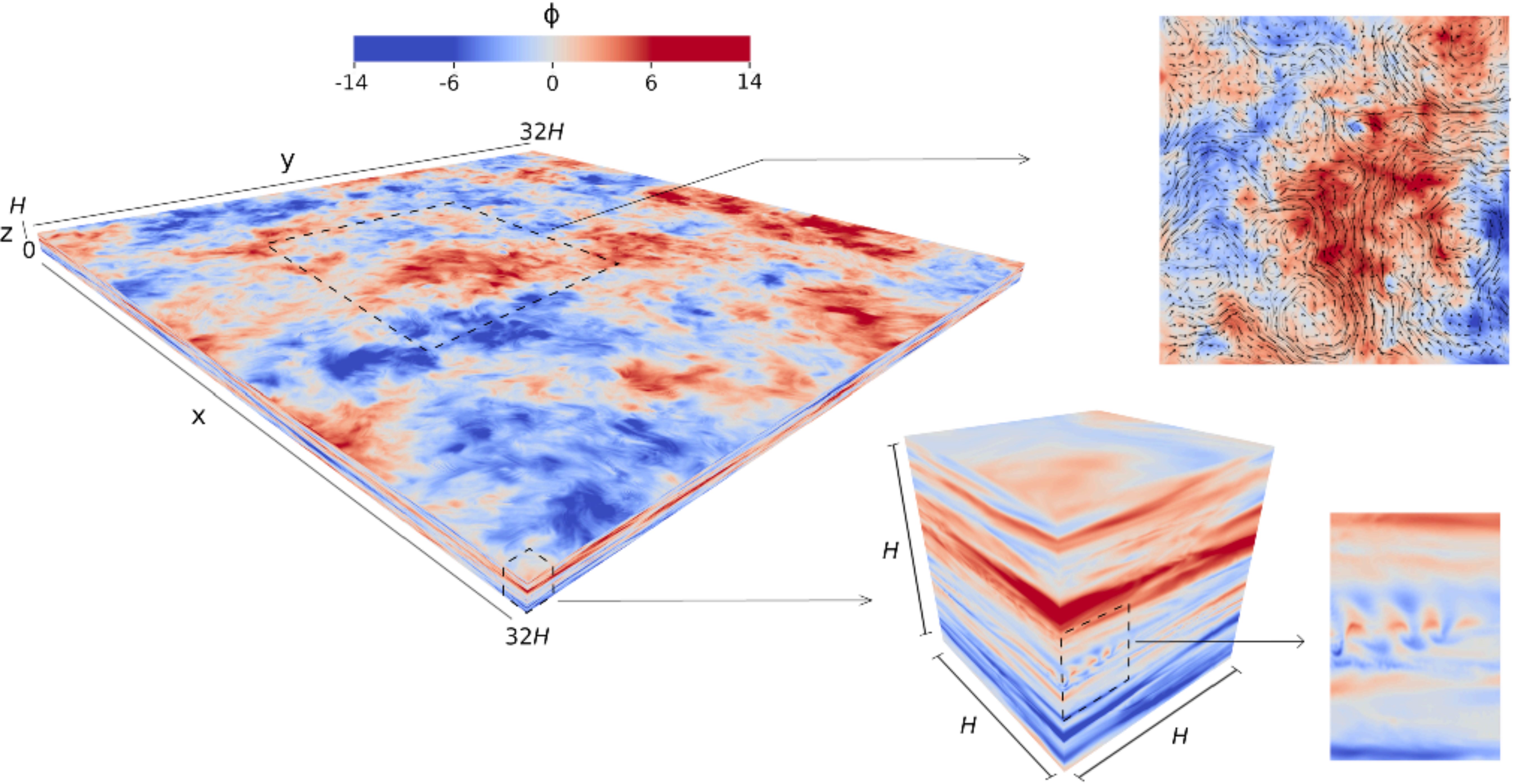
$$\boxed{D_g \zeta = f_0 \frac{\partial w_1}{\partial z}} \quad \text{with} \quad \zeta = f_0 + \beta y + \nabla_\perp^2 \psi$$

Absolute vorticity

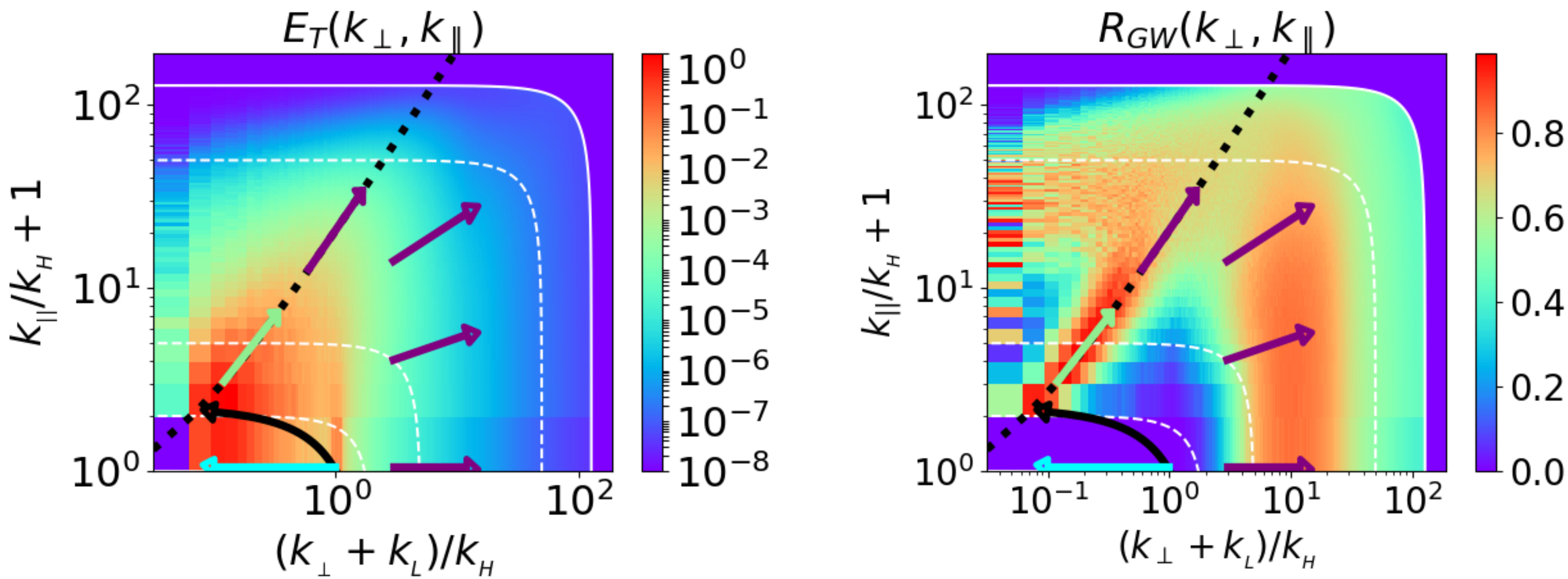
And using $w_1 = D_g \left(\frac{f_0}{N^2} \frac{\partial \psi}{\partial z} \right)$

$$\Rightarrow \quad \boxed{D_g q = 0} \quad \text{with} \quad q = \zeta + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right)$$

QG potential vorticity



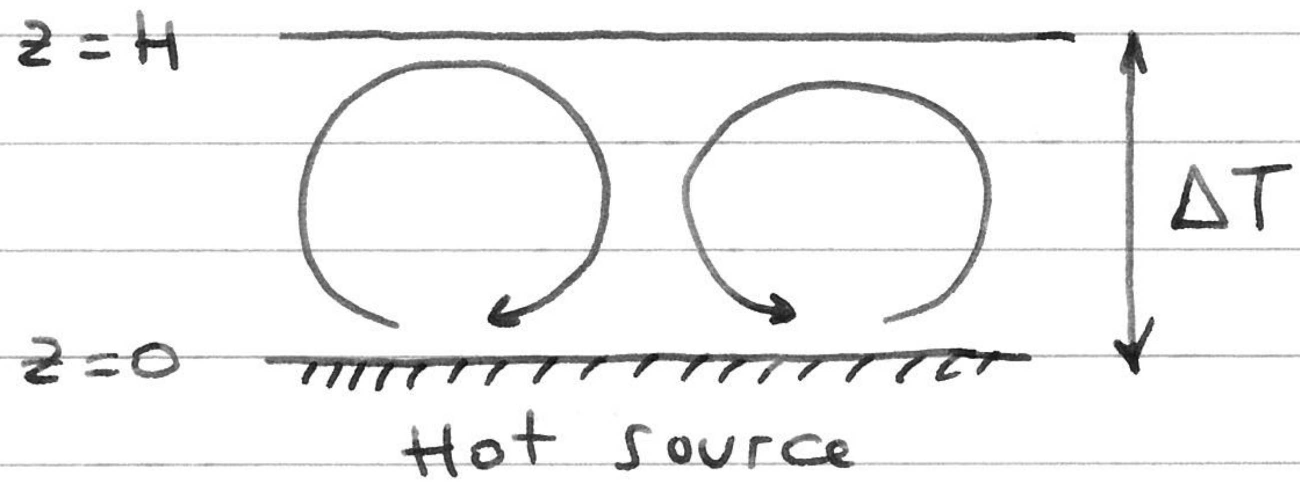
Alexakis et al. (2024)

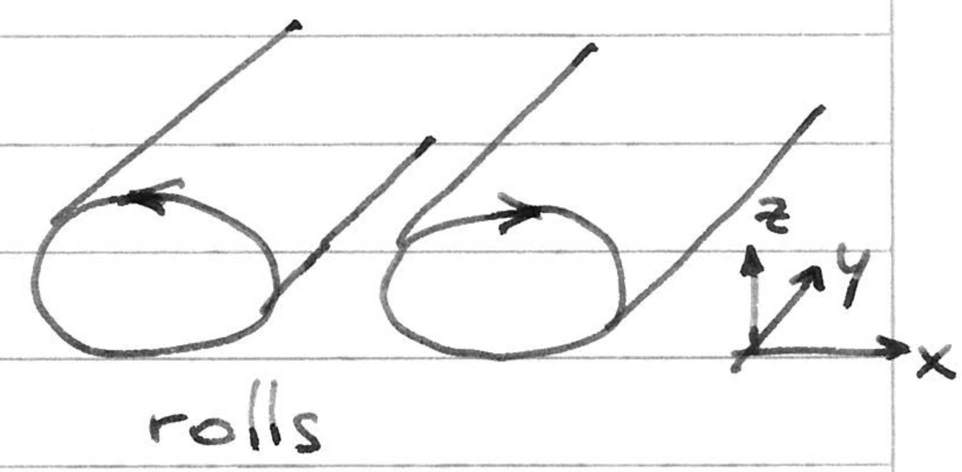


$E_T(k_\perp, k_\parallel)$
$R_{GW}(k_\perp, k_\parallel)$
$k_\parallel/k_H + 1$
$(k_\perp + k_L)/k_H$
10^0
10^1
10^2
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0.0
0.2
0.4
0.6
0.8

A diversion: The Lorenz attractor

When the flow is unstable, some modes grow exponentially until non-linear saturation. Let's assume this set up:





In the 2D case we can write $u = -\dfrac{\partial \psi}{\partial z}$, $w = \dfrac{\partial \psi}{\partial x}$

$$\psi = \frac{(1+a^2)k\sqrt{2}}{a} X \sin(\pi a \xi) \sin(\pi \zeta)$$

$$\varphi = \frac{R_c \Delta T}{\pi R_a} \left[\sqrt{2}\, Y \cos(\pi a \xi) \sin(\pi \zeta) - Z \sin(2\pi \zeta) \right]$$

with $$\varphi = \sqrt{-\frac{g}{\rho_0}\left(\frac{\partial \bar{\rho}}{\partial z}\right)^{-1}}\ \rho$$

thermal exp. coefficient

$$\xi = \frac{x}{H} \qquad \zeta = \frac{z}{H} \qquad R_a = \frac{g \alpha H^3 \Delta T}{\nu K}$$

$$\tau = \pi^2 (1+a^2) kt / H^2$$

thermal diffusivity

This is a Galerkin truncation.

Replacing in the Boussinesq equations

$$\begin{cases} \dot{X} = & -\sigma X + \sigma Y \\ \dot{Y} = -XZ & + rX - Y \\ \dot{Z} = XY & - bZ \end{cases}$$

with $r = \dfrac{R_a}{R_c}$ $\quad b = \dfrac{4}{1+a^2}$ $\quad \sigma = \dfrac{\nu}{K}$

finally

$$X = X' , \quad Y = Y' , \quad Z = Z' + \sigma + r .$$

$$\Rightarrow \begin{cases} \dot{X}' = \sigma\left(Y' - X'\right) \\ \dot{Y}' = -X'Z' - \sigma X' - Y' \\ \dot{Z}' = X'Y' - bZ' - b(\sigma + r) \end{cases}$$

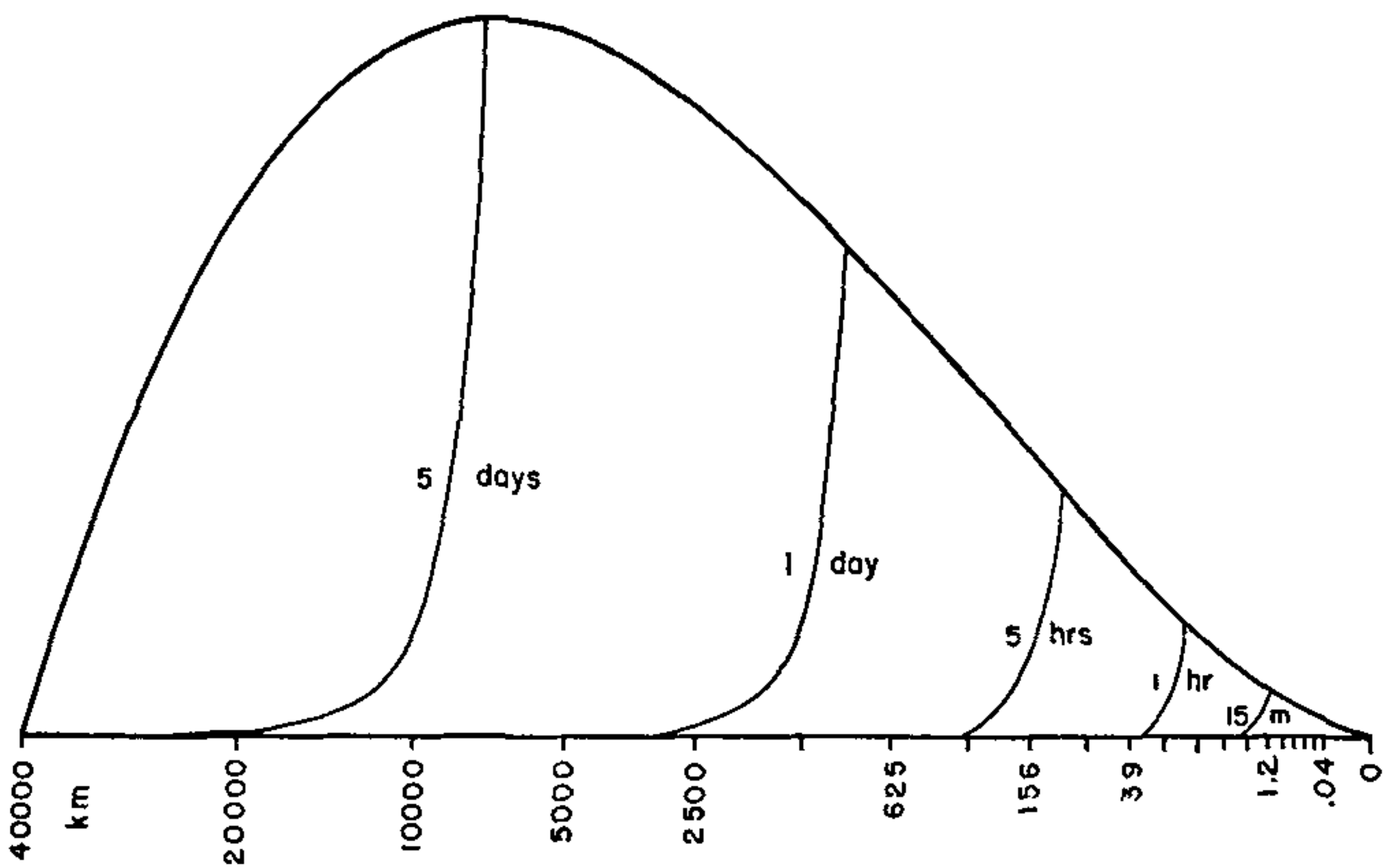


Fig. 2. Basic energy spectrum (heavy curve), and error-energy spectra (thin curves) at 15 minutes, 1 hour, 5 hours, 1 day, and 5 days, as interpolated from numerical solution in Experiment A. Thin curves coincide with heavy curve, to the right of their intersections with heavy curve. Horizontal coordinate is fourth root of wave length, labeled according to wave length. Resolution intervals are separated by vertical marks at base of diagram. Vertical coordinate is energy per unit logarithm of wave length, divided by fourth root of wave length. Areas are proportional to energy.

Lorenz (1969): Not from the strange attractor but from turbulence.