An introduction to geophysical fluid dynamics

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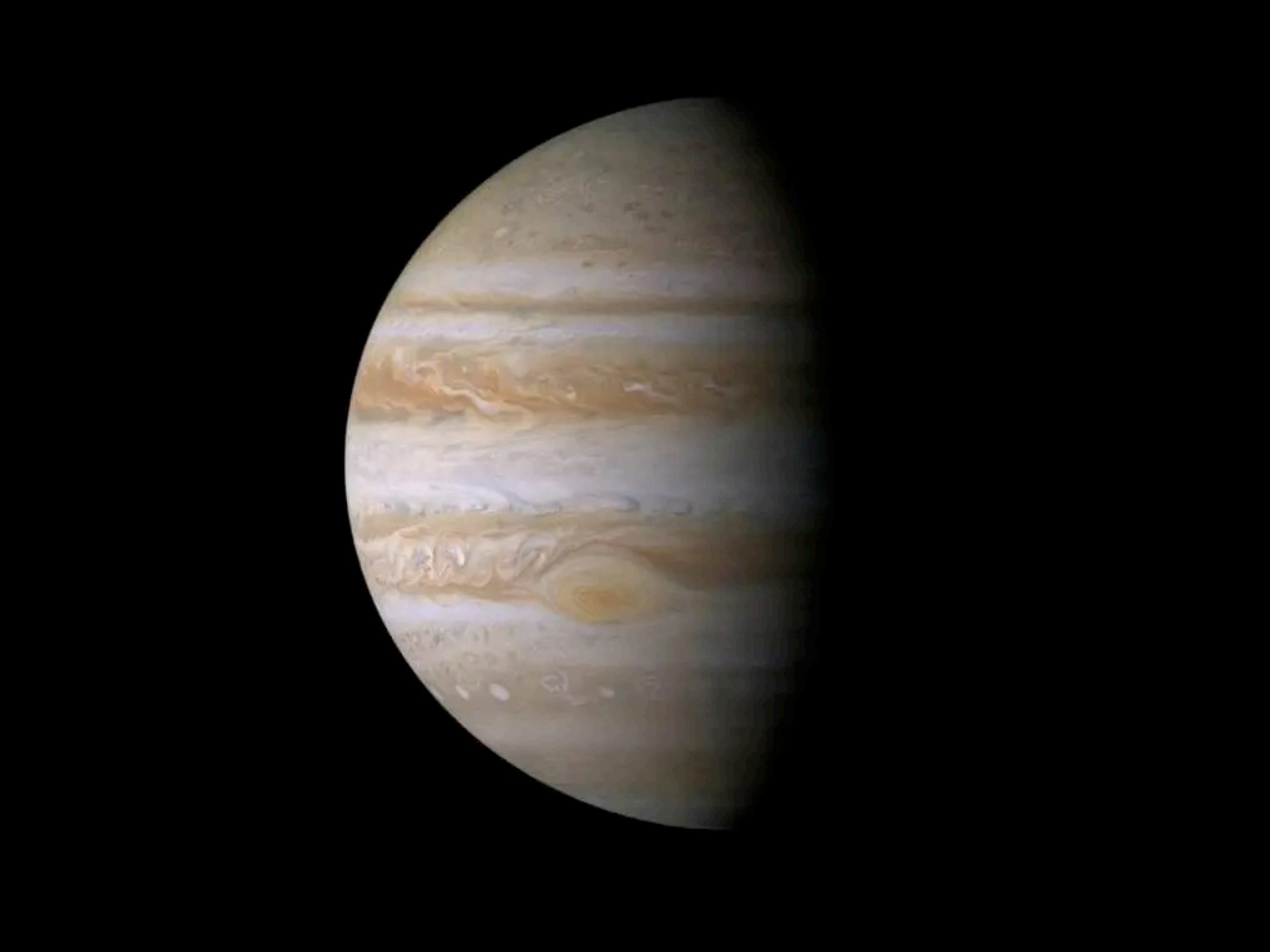
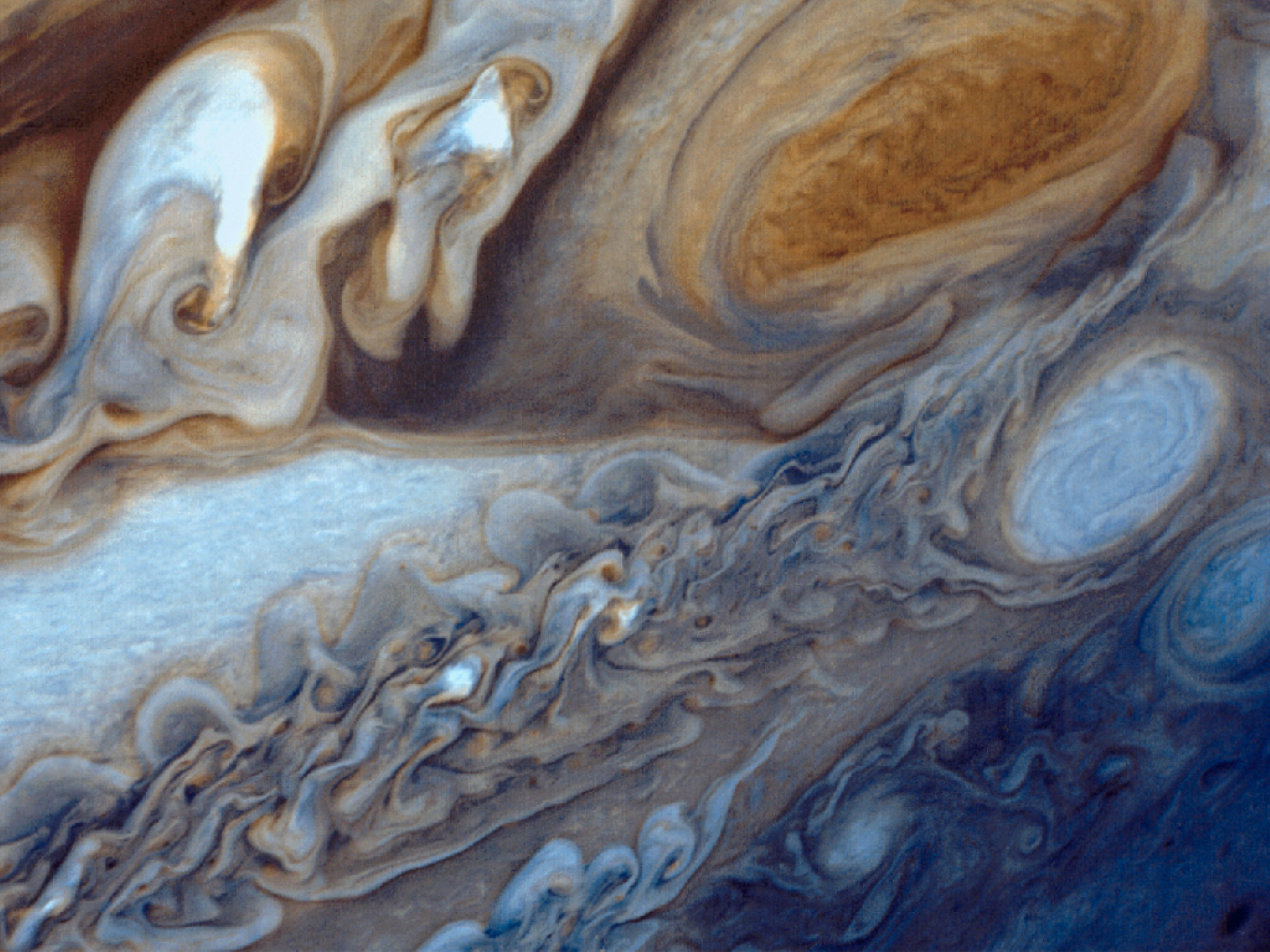
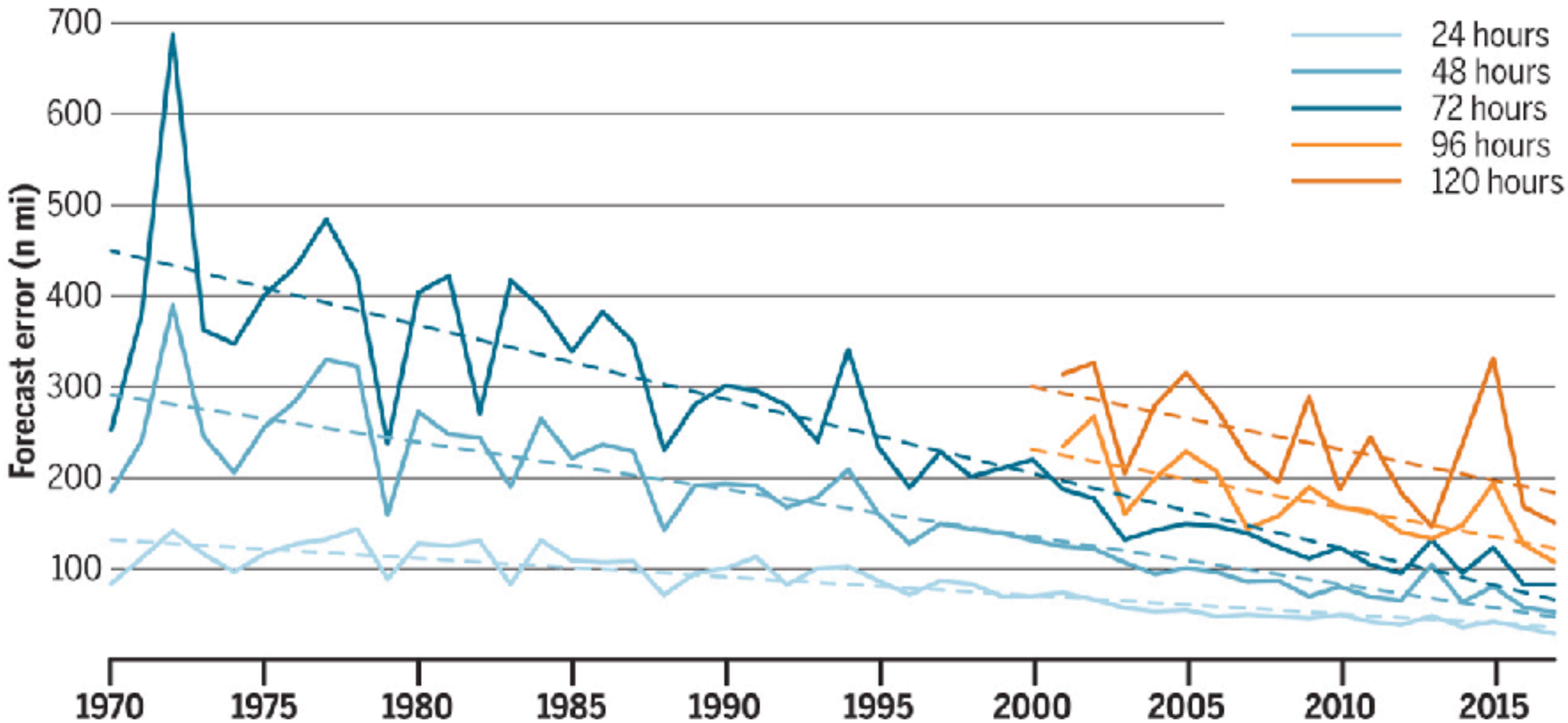




Advances in hurricane prediction

Data from the NOAA National Hurricane Center (NHC) (*13*) show that forecast errors for tropical storms and hurricanes in the Atlantic basin have fallen rapidly in recent decades. The graph shows the forecast error in nautical miles (1 n mi = 1.852 km) for a range of time intervals.



Alley et al., Science (2019)

Basic physical processes → formal models

forecast ← simplification

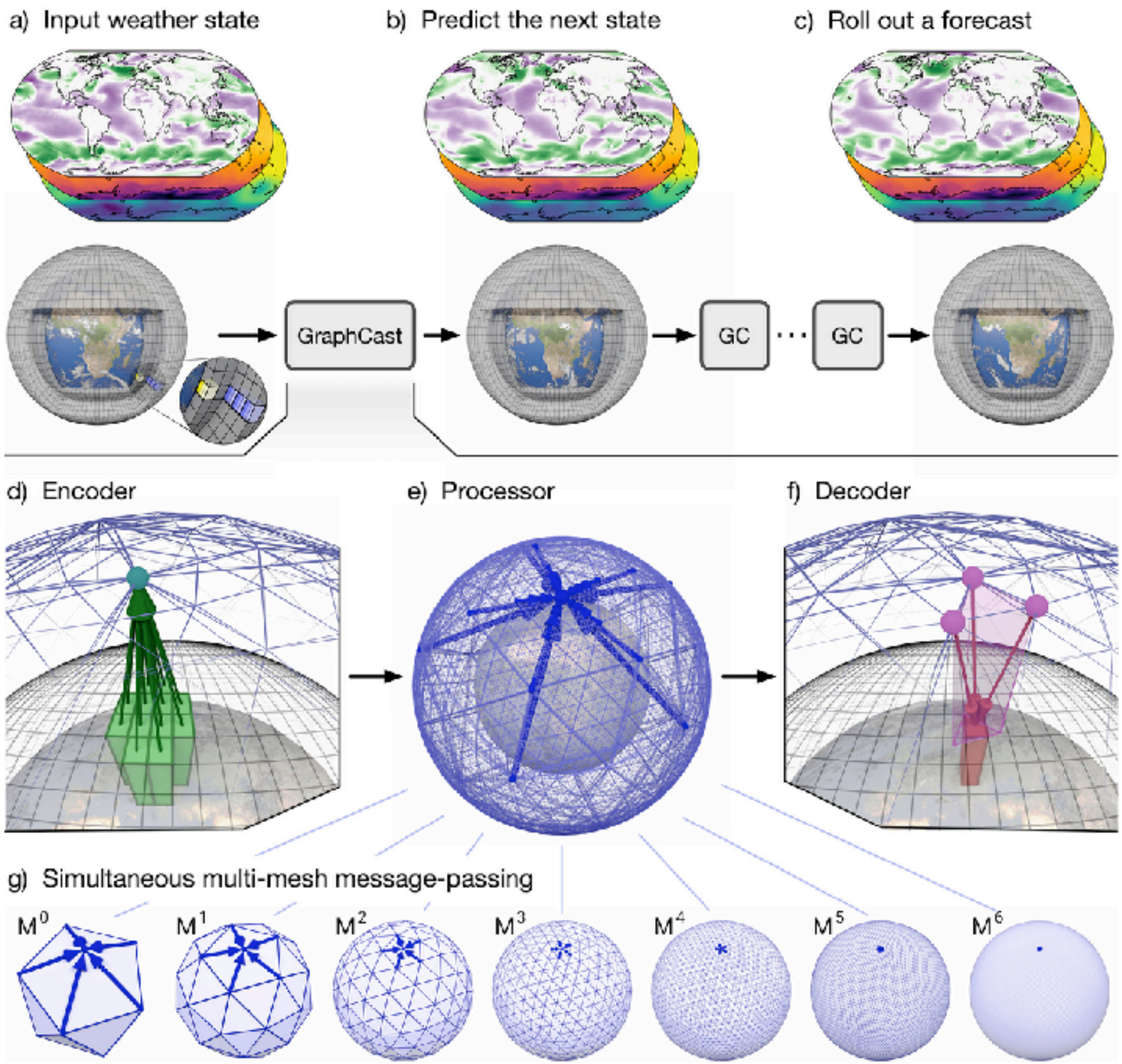
NN? Models vs. data?

Or models + data

Physics → conservation laws

Physics → forces

Physics → Geometry and B.C.

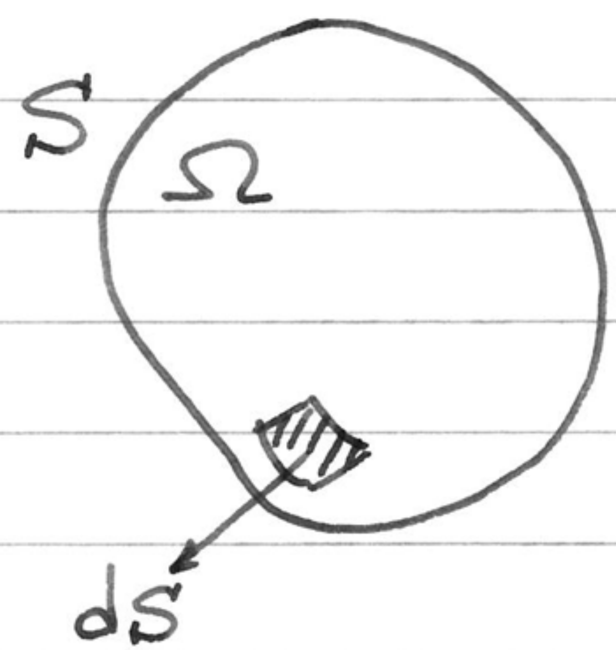


Lam et al., Science (2023)

The fundamental equations

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \\
\dfrac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\dfrac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{v} + \left(\dfrac{\nu}{3} + \nu' \right) \underline{\nabla} (\underline{\nabla} \cdot \underline{v}) + \dfrac{\underline{f}}{\rho} \\
\dfrac{\partial \theta}{\partial t} + \underline{v} \cdot \underline{\nabla} \theta = S'
\end{cases}
$$

Mass conservation



$$\frac{\partial}{\partial t}\int \rho \, dV = -\int \rho \underline{v} \cdot d\underline{S} =$$

$$-\int \underline{\nabla} \cdot (\rho \underline{v}) \, dV$$

$$\Longrightarrow \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0}$$

Then the conservation of mass is

$$\boxed{\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{v} = 0}$$

and in an incompressible fluid $\boxed{\underline{\nabla} \cdot \underline{v} = 0}$

<u>Example</u>: Ideal gas in the atmosphere

$$P = \rho R T$$

Atmospheric pressure $\approx 10^5$ Pa
δP from meteorological phenomena $\approx 10^3$ Pa
Sound waves (60 dB) ≈ 10 Pa

Conservation of momentum

$$\rho \frac{D\underline{u}}{Dt} = \underline{F}$$

pressure gradients
viscosity
gravity
rotation
external forces

$$//$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla} u\right) = \underline{F}$$

Energy

$$\frac{Ds}{Dt} = 0$$

If we have sources (e.g., latent heat)

$$\frac{\partial s}{\partial t} + \underline{u}\cdot\underline{\nabla} s = S$$

For an ideal gas

$$S = C_p \ln T - Nk \ln P + S_0$$

$$= C_p \ln\left(\frac{T}{P^K}\right) + S_0 \qquad \text{with} \quad K = \frac{Nk}{C_p}$$

If fluid elements are adiabatic $dS = 0 = C_p \frac{dT}{T} - Nk \frac{dP}{P}$

$$C_p \ln\left(\frac{\Theta}{T}\right) = Nk \ln\left(\frac{P_0}{P}\right)$$

$$\Rightarrow \boxed{\Theta = T\left(\frac{P_0}{P}\right)^K}$$

Potential temperature

Using $\quad S = C_p \ln \Theta + S_1$

$$\Rightarrow \boxed{\frac{\partial \Theta}{\partial \Theta} + \underline{v} \cdot \underline{\nabla} \Theta = S'}$$

Vorticity

$$\underline{\omega} = \underline{\nabla} \times \underline{v}$$

For an incompressible flow (but ρ may not be constant)

$$\underline{\nabla} \times \frac{\partial \underline{v}}{\partial t} = \frac{\partial \underline{\omega}}{\partial t} = - \underline{\nabla} \times \left(\underline{v} \cdot \underline{\nabla} \underline{v} \right) - \underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} p \right)$$
$$+ \underline{\nabla} \times \left(\nu \nabla^2 \underline{v} \right) + \underline{\nabla} \times \left(\frac{\underline{f}}{\rho} \right)$$

Using
$$\begin{cases} \underline{v} \cdot \underline{\nabla} \underline{v} = - \underline{v} \times \underline{\omega} + \nabla^2 \left(\frac{v^2}{2} \right) \\ \nabla \times \left(\underline{v} \times \underline{\omega} \right) = - \cancel{\underline{\omega} \left(\underline{\nabla} \cdot \underline{v} \right)} + \left(\underline{\omega} \cdot \underline{\nabla} \right) \underline{v} - \left(\underline{v} \cdot \underline{\nabla} \right) \underline{\omega} \\ \underline{\nabla} \times \left(\frac{1}{\rho} \underline{\nabla} p \right) = - \frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} p \end{cases}$$

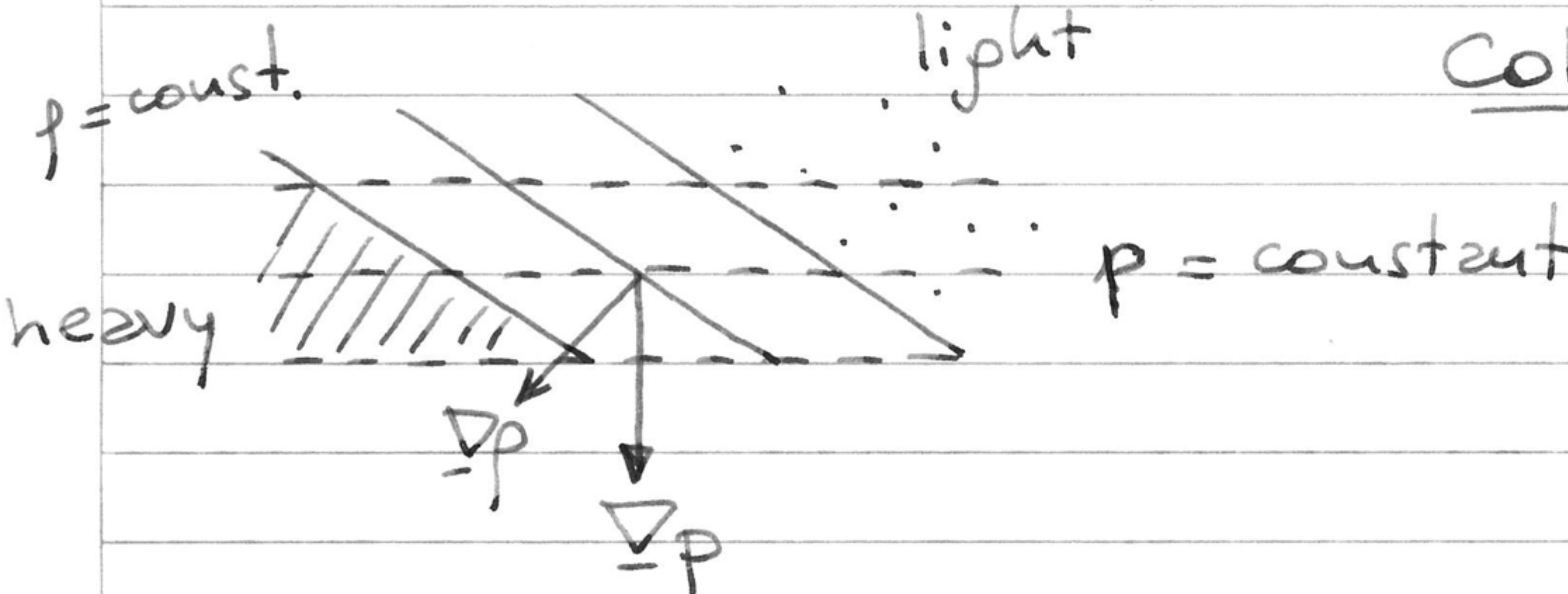
$$\Rightarrow \overbrace{\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \underline{\nabla}\, \underline{\omega}}^{\frac{D\underline{\omega}}{Dt}} = \underline{\omega} \cdot \underline{\nabla}\, \underline{u} + \overbrace{\frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} p}^{\text{baroclinic term}}$$

$$+ \nabla \times \left(\nu \nabla^2 \underline{u} \right) + \underline{\nabla} \times \left(\frac{\underline{f}}{\rho} \right)$$

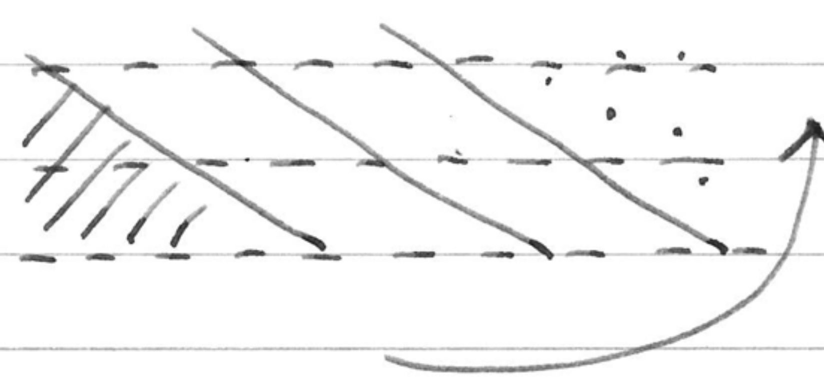
If the flow is <u>barotropic</u>, $\rho = \rho(p) \Rightarrow \underline{\nabla} \rho \times \underline{\nabla} p = 0$

Example : generation of circulation by baroclinicity

Cold front



Light and heavy fluid feel the same force ($-\underline{\nabla}p$), but light fluid moves upwards faster generating circulation



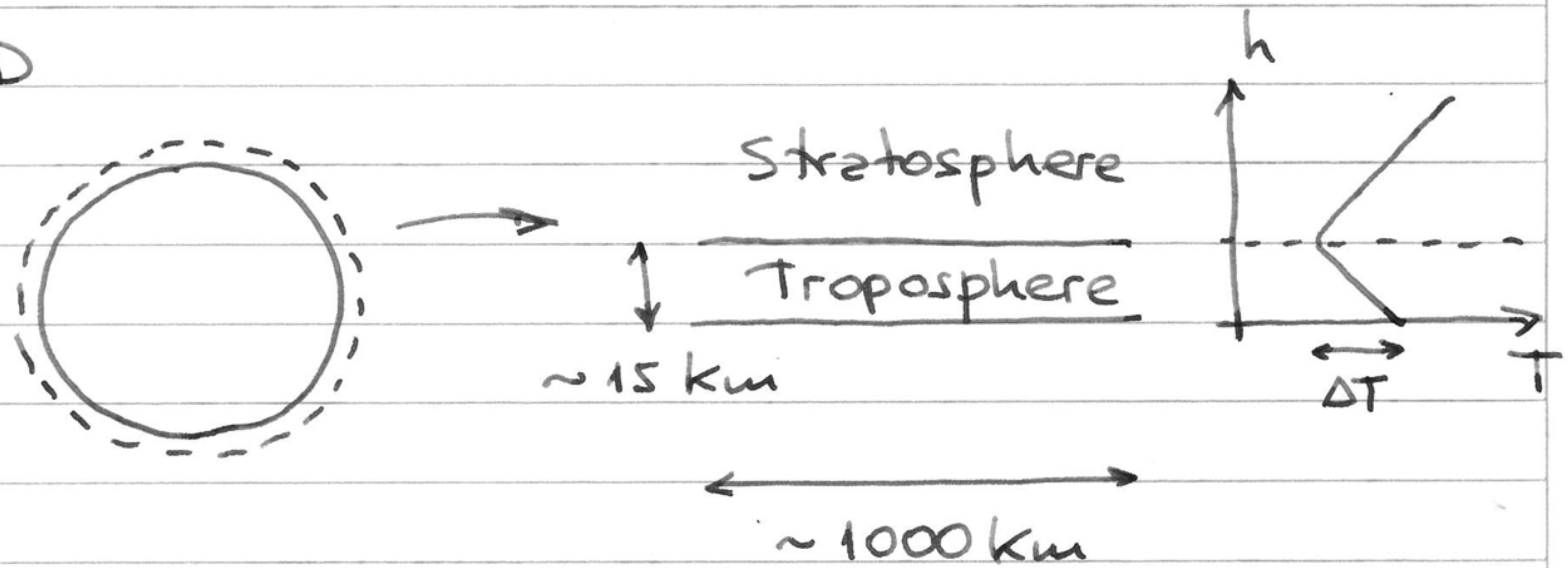
$\underline{\nabla}\rho$ and $\underline{\dot{\nabla}}p$
try to align!

If $\rho =$ const. ($\nu =$ const.) and $\underline{f}$ conservative

$$\Rightarrow \boxed{\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} + \nu\nabla^2\underline{\omega}}$$

vortex stretching

In 2D

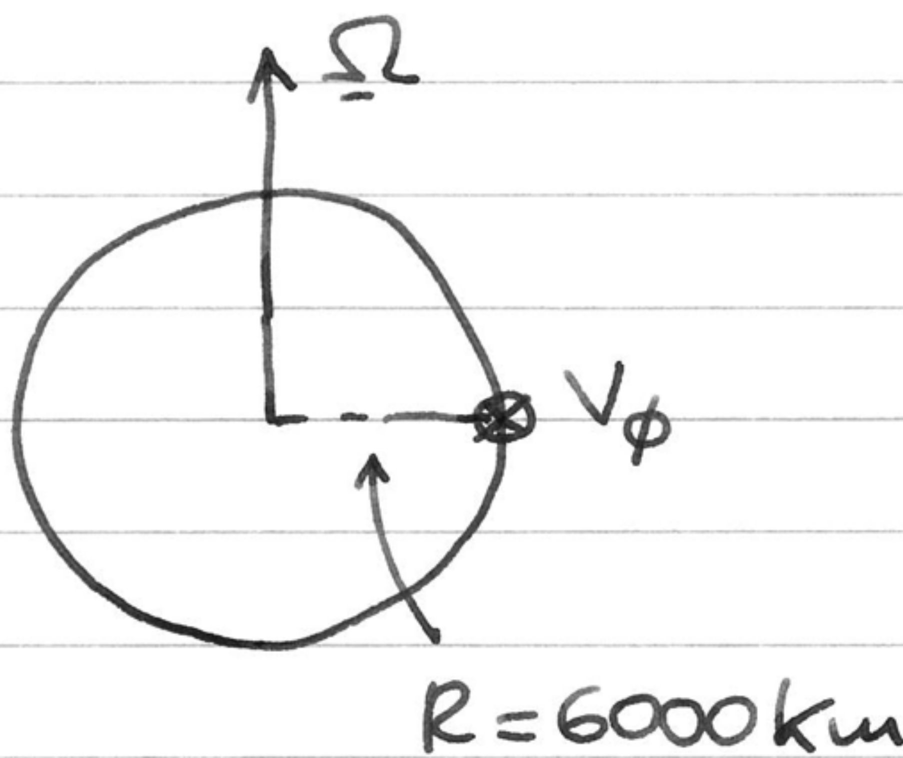


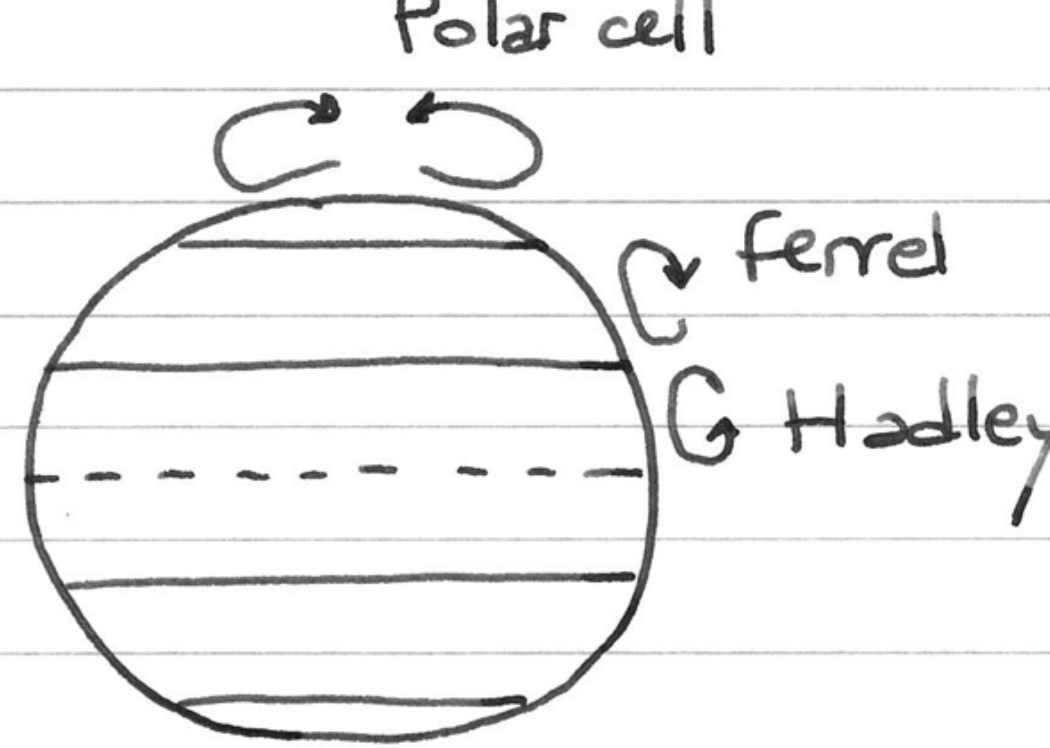
$$\underline{v} = (v_x, v_y, 0) = \underline{v}(x, y)$$

$$\underline{\omega} = \underline{\nabla}\times\underline{v} = \omega\hat{z}$$

$$\Rightarrow \boxed{\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}}$$

Rotating flows





$$T = \frac{2\pi}{\Omega} = 24\ \mathrm{h}$$

$$V_\phi = \Omega R \approx 1600\ \mathrm{km/h}$$

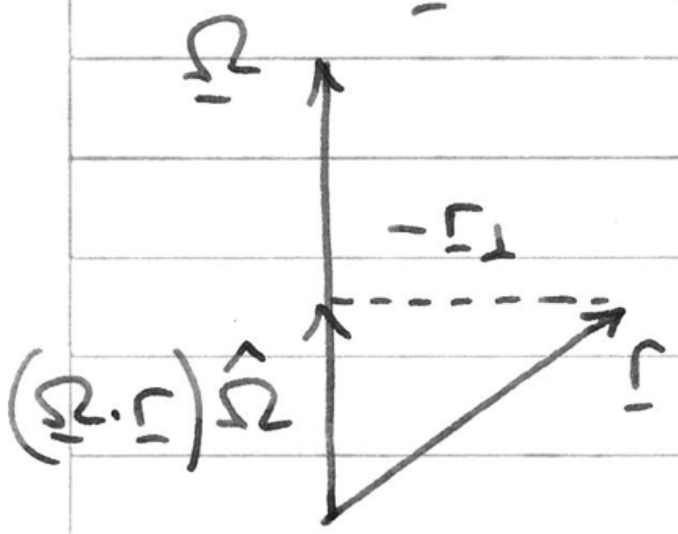
Atmospheric phenomena has $v \lesssim 200\ \mathrm{km/h}$
(jet currents)

$\Rightarrow$ It makes sense to work in the rotating frame!

Coriolis, centrifugal force

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\underline{v}\right) = \underbrace{-\rho\, 2\,\underline{\Omega}\times\underline{v}}_{\text{Coriolis}} \underbrace{-\,\rho\,\underline{\Omega}\times(\underline{\Omega}\times\underline{r})}_{\text{centrifugal force}}$$
$$-\underline{\nabla}p + \mu\nabla^2\underline{v} + \underline{f}$$

The centrifugal force can be absorbed into $\underline{f}$ or p:



$$\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = (\underline{\Omega}\cdot\underline{r})\,\underline{\Omega} - \Omega^2\underline{r} =$$
$$= \Omega^2\left[(\hat{\underline{\Omega}}\cdot\underline{r})\hat{\underline{\Omega}} - \underline{r}\right] = -\Omega^2\underline{r}_\perp$$
$$\Rightarrow -\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = \Omega^2\underline{r}_\perp = -\nabla\varphi_{ce}$$

with $\varphi_{ce} = -\dfrac{\Omega r_\perp^2}{2}$

$$\Rightarrow \boxed{\frac{D\underline{v}}{Dt} = -2\underline{\Omega} \times \underline{v} - \frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{v} + \frac{\underline{f}}{\rho} - \underline{\nabla}\varphi_{ce}}$$

It corrects the effective gravity

In the Earth the correction is $\mathcal{O}(10^{-3})$

Dimensionless numbers

Taking

$$\frac{|\underline{v}\cdot\nabla\underline{v}|}{|\nu\nabla^2\underline{v}|} \sim \frac{U^2}{L}\cdot\frac{L^2}{U\nu} \sim \boxed{\frac{UL}{\nu} = Re}$$

$$\frac{|\underline{v}\cdot\nabla\underline{v}|}{|2\Omega\times\underline{v}|} \sim \frac{U^2}{L}\cdot\frac{1}{2\Omega U} \sim \boxed{\frac{U}{2L\Omega} = R_o}$$

In the Earth ($L \sim 100-1000\,km$) $R_o \approx 0.1$

In Jupiter red spot $R_o \approx 0.015$

Taylor-Proudman theorem

$\nu = \underline{f} = 0$, using $\underline{u}\cdot\underline{\nabla}\,\underline{u} = -\underline{u}\times\underline{\omega} + \nabla^2\left(\frac{u^2}{2}\right)$

$$\Rightarrow \frac{\partial \underline{u}}{\partial t} + \big(\overbrace{\underline{\omega} + 2\underline{\Omega}}^{\underline{\omega}_a}\big)\times\underline{u} = -\frac{1}{\rho}\underline{\nabla}\, p$$

Taking the curl and using

$$\nabla\times\left(\underline{\omega}_a\times\underline{u}\right) = \cancel{\underline{\omega}_a\,\underline{\nabla}\cdot\underline{u}} - \left(\underline{\omega}_a\cdot\underline{\nabla}\right)\underline{u} + \left(\underline{u}\cdot\underline{\nabla}\right)\underline{\omega}_a$$

$$\Rightarrow \frac{D\underline{\omega}}{Dt} = \left(\underline{\omega}_a\cdot\nabla\right)\underline{u} - \frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla}p$$

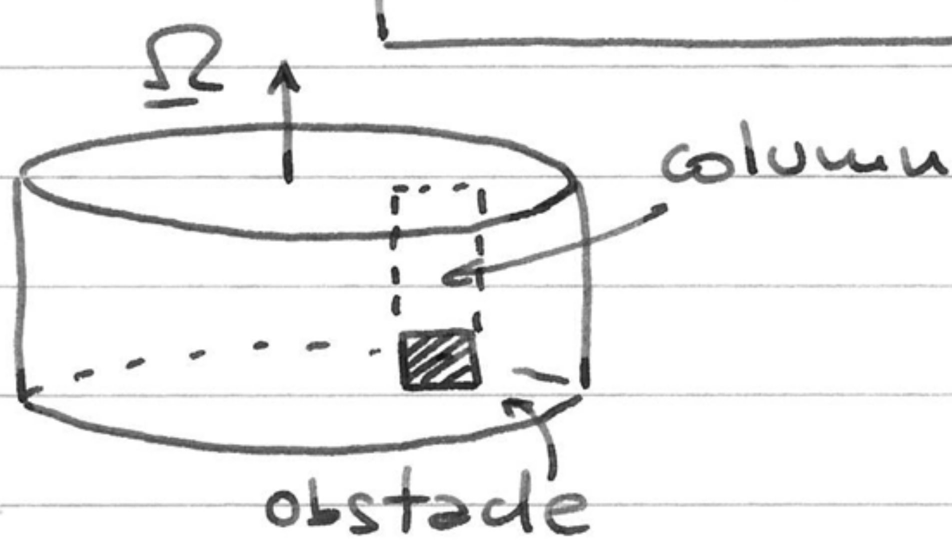
Let's consider a barotropic stationary flow with $R_o \ll 1 \quad \Rightarrow \quad |\underline{\omega}| \ll |\underline{\Omega}|$ and $\underline{\omega}_a \approx 2\underline{\Omega}$

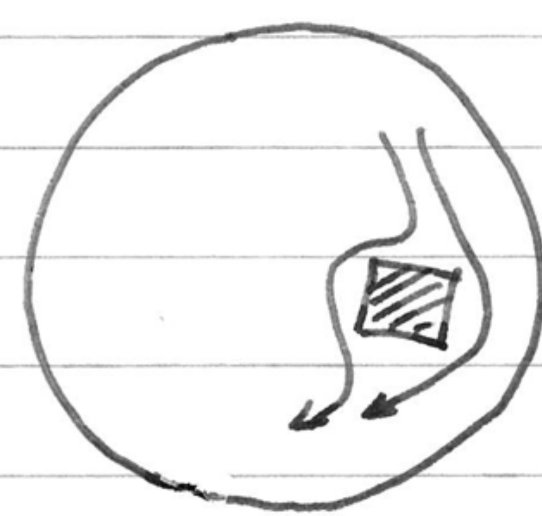
$$\Rightarrow \qquad (2\underline{\Omega} \cdot \underline{\nabla})\,\underline{v} = 0$$

For $\Omega = \Omega \hat{z}$, $\underline{v} = (u, v, w)$

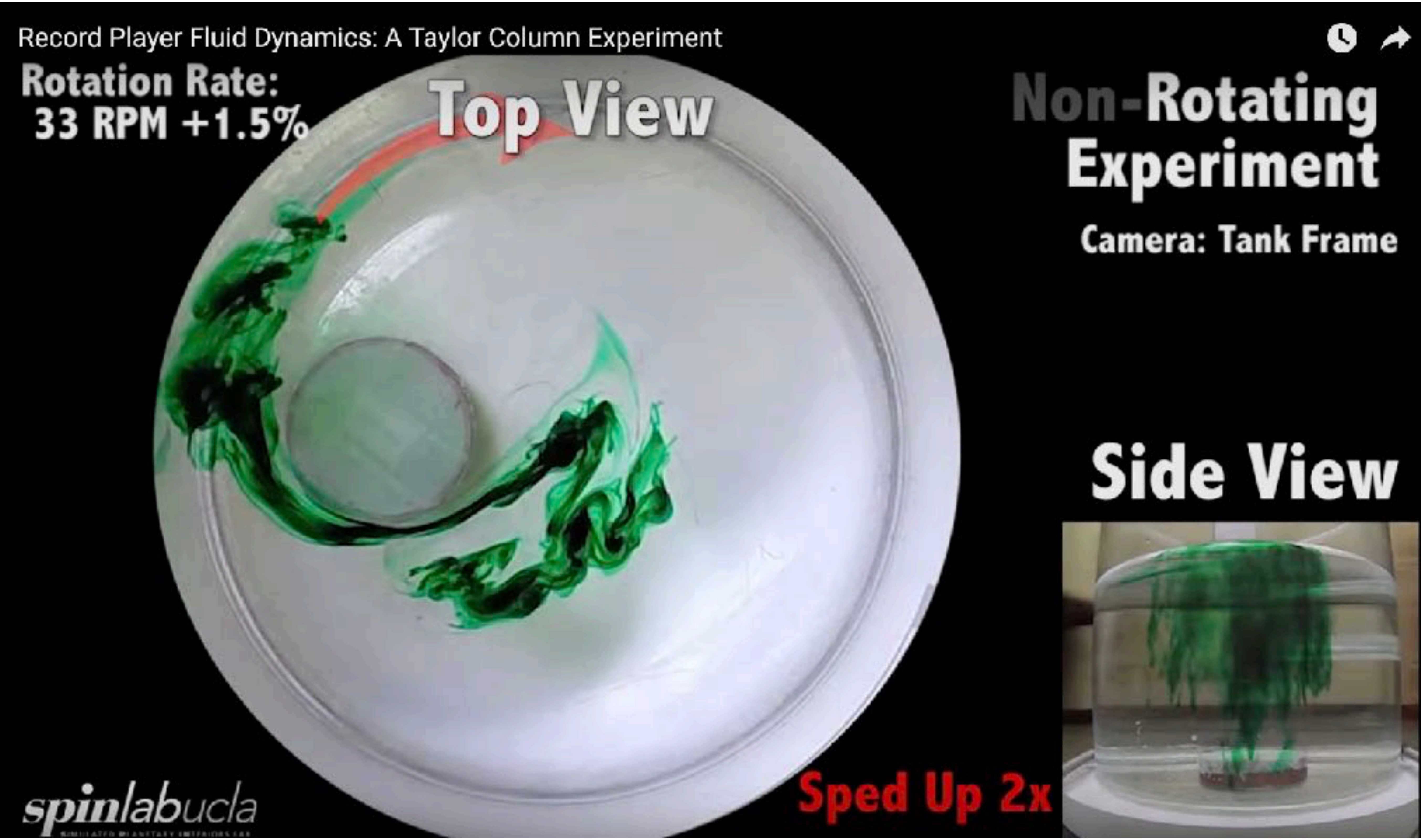
$$\Rightarrow \begin{cases} 2\Omega\, \partial_z u = 0 \\ 2\Omega\, \partial_z v = 0 \\ 2\Omega\, \partial_z w = 0 \end{cases}$$

$$\Rightarrow \quad \boxed{\underline{v} = \underline{v}(x, y)}$$

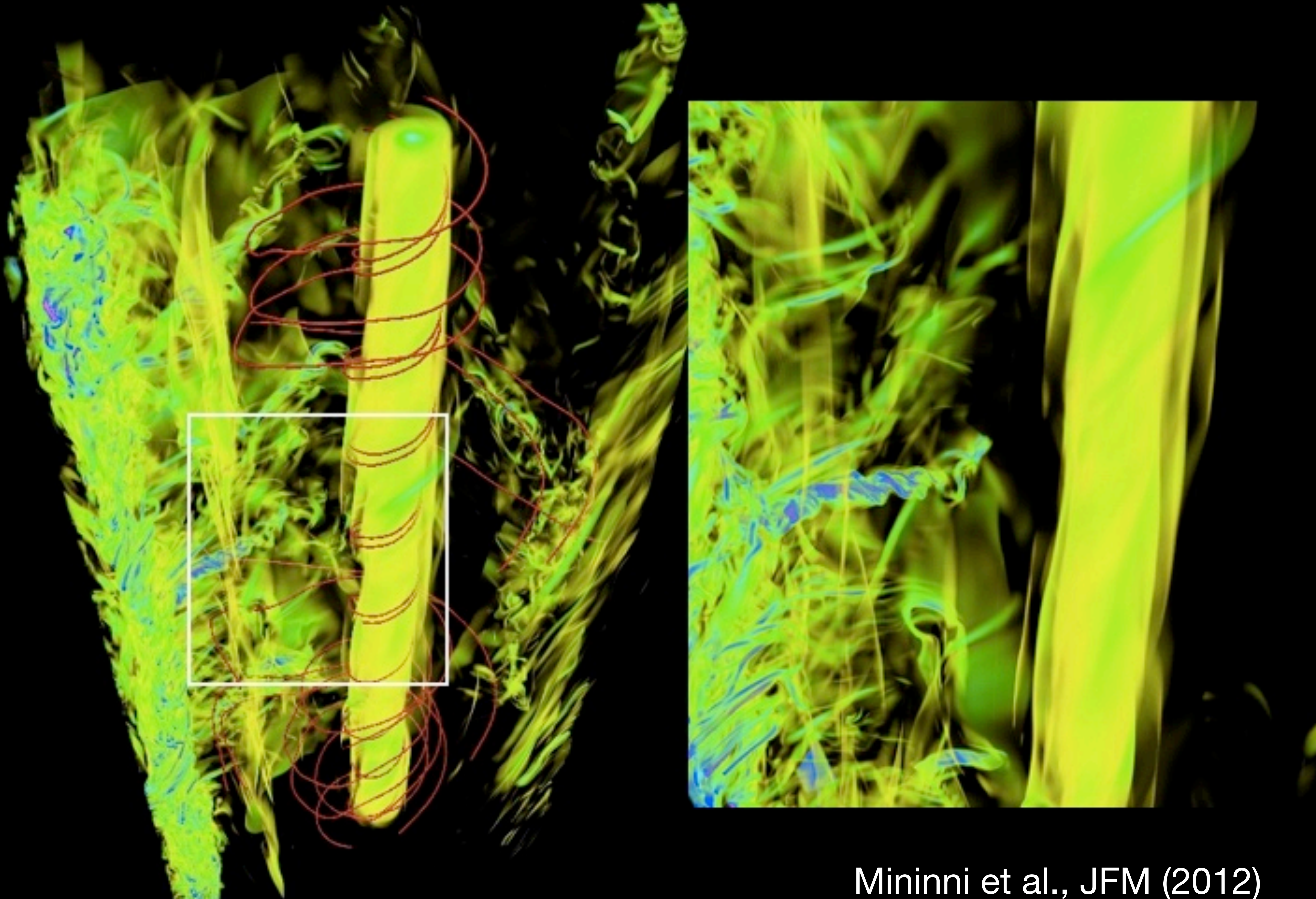




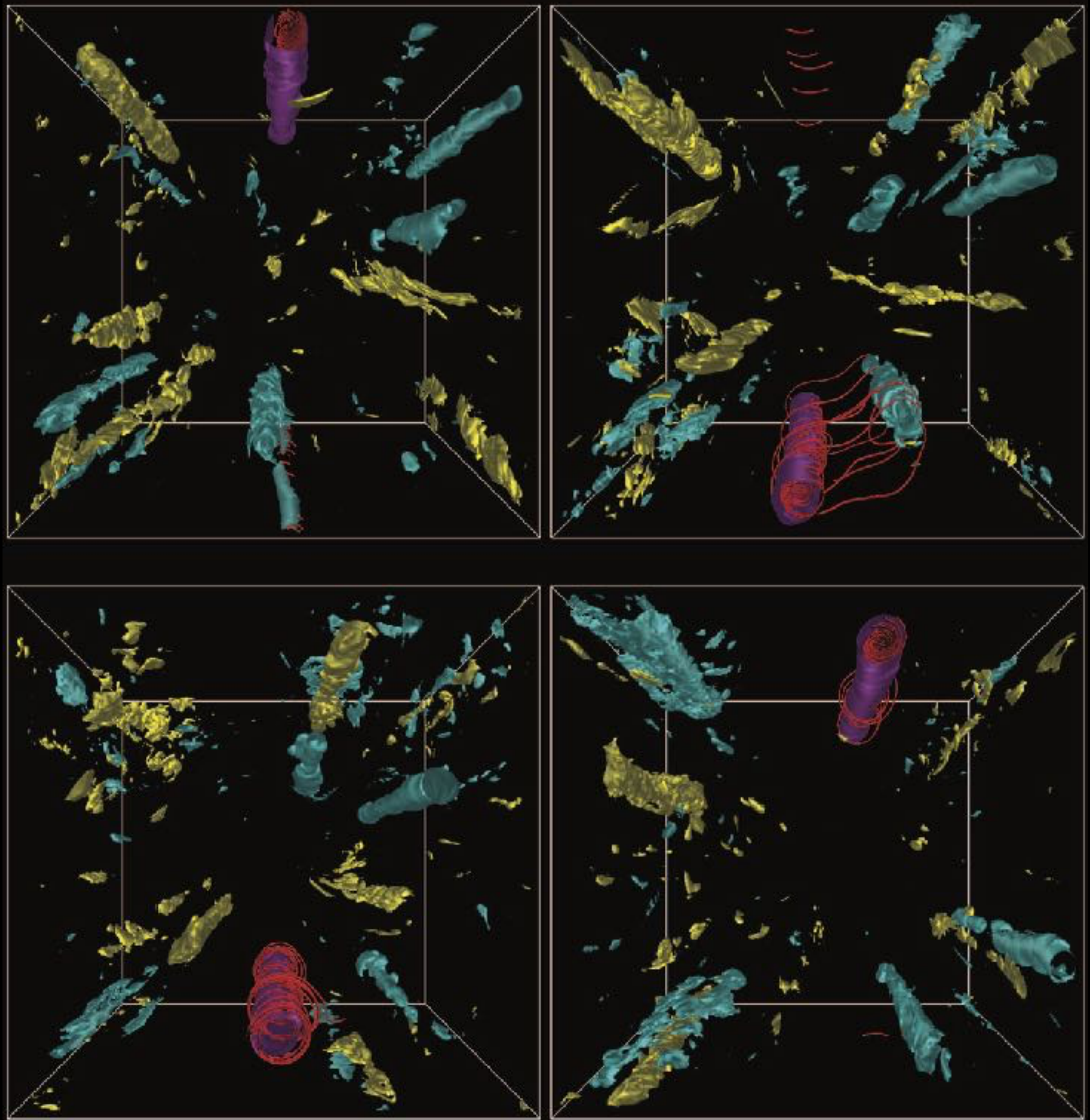
blocking!

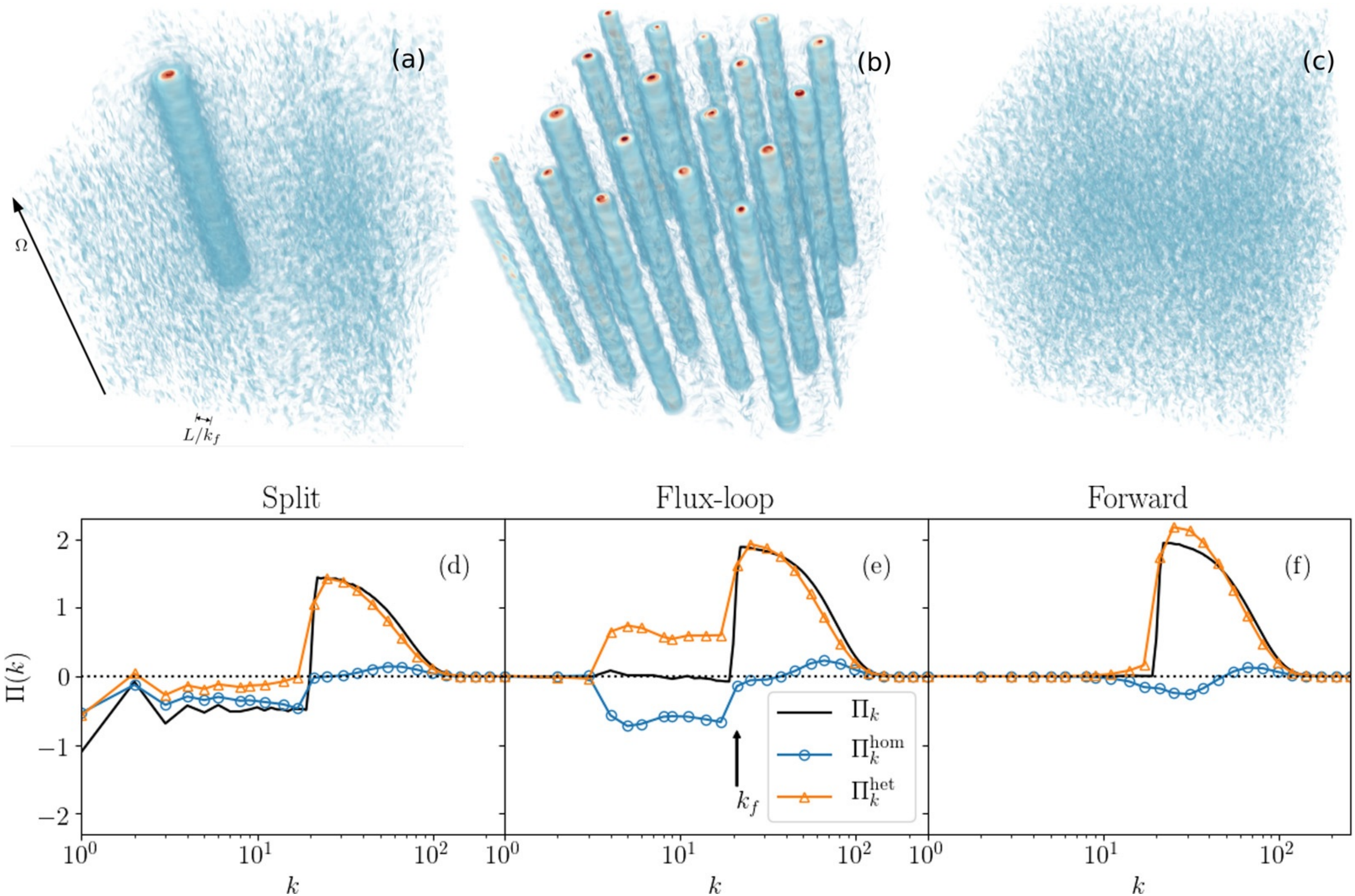


https://www.youtube.com/watch?v=7GGfsW7gOLI

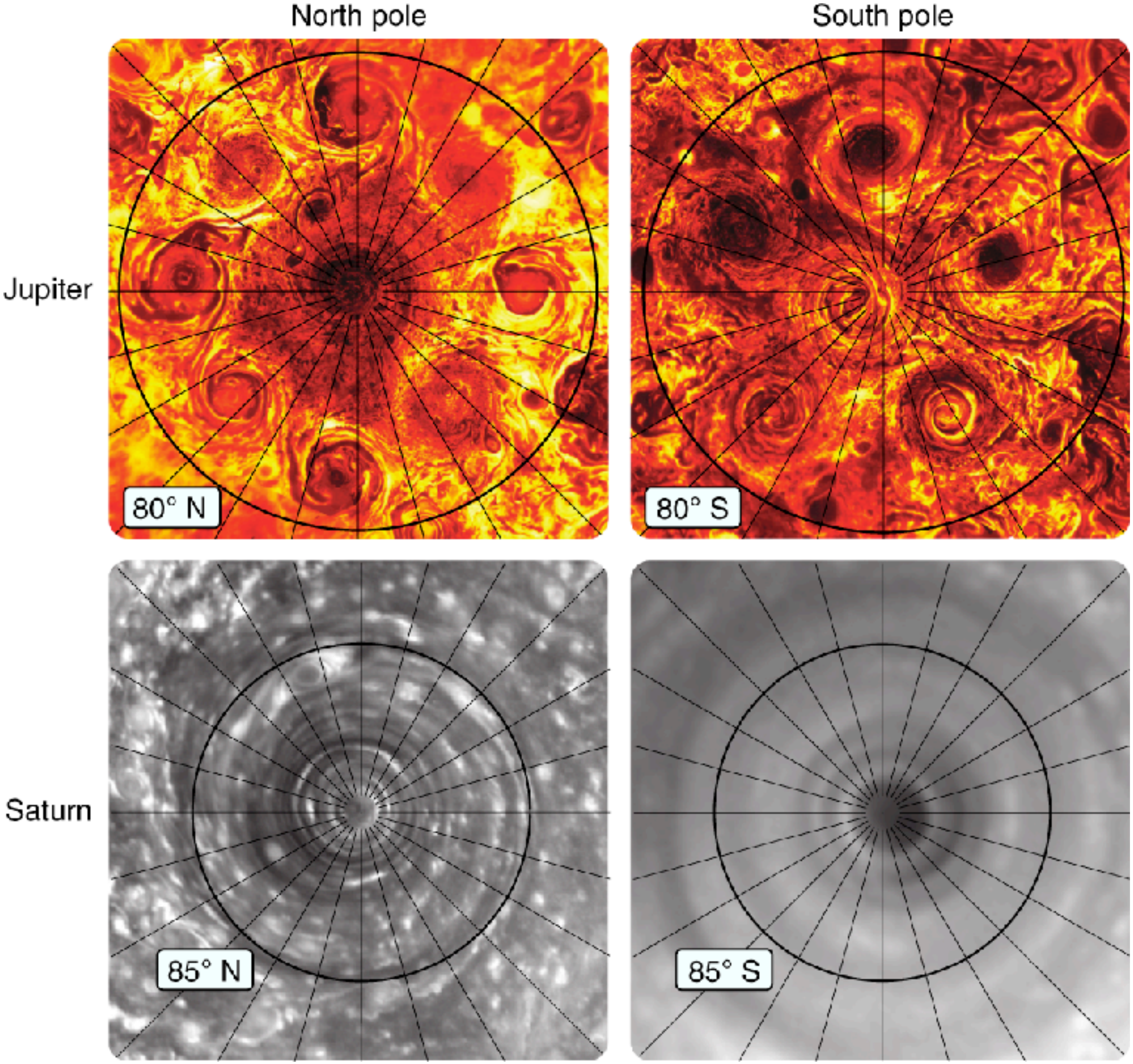


Mininni et al., JFM (2012)





Clark di Leoni, Alexakis, Biferale & Buzzicotti, PRFluids (2020)



Siegelman et al., Nat. Physics 2022)

Inertial waves

Let's consider $\nu = \underline{f} = 0$, $R_o \ll 1$, barotropic flow

$$\Rightarrow \frac{\partial \underline{\omega}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega}_a + \underline{\nabla} \underline{v}$$

Linearizing for $|\underline{\omega}|, |\underline{v}| \ll 1$

$$\frac{\partial \underline{\omega}}{\partial t} = 2 \underline{\Omega} \cdot \nabla \underline{v}$$

Taking $\nabla \times \frac{\partial}{\partial t}$, and using $\underline{\nabla} \times \underline{\nabla} \times \underline{v} = -\nabla^2 \underline{v} + \cancel{\underline{\nabla}(\underline{\nabla} \cdot \underline{v})}$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2}\left(\nabla^2 \underline{v}\right) + 4\left(\underline{\Omega} \cdot \underline{\nabla}\right)^2 \underline{v} = 0}$$

It looks like a wave equation.

We look for solutions

$$\underline{u} = \underline{u}_0 \, e^{i(\underline{k}\cdot\underline{x} - \sigma t)}$$

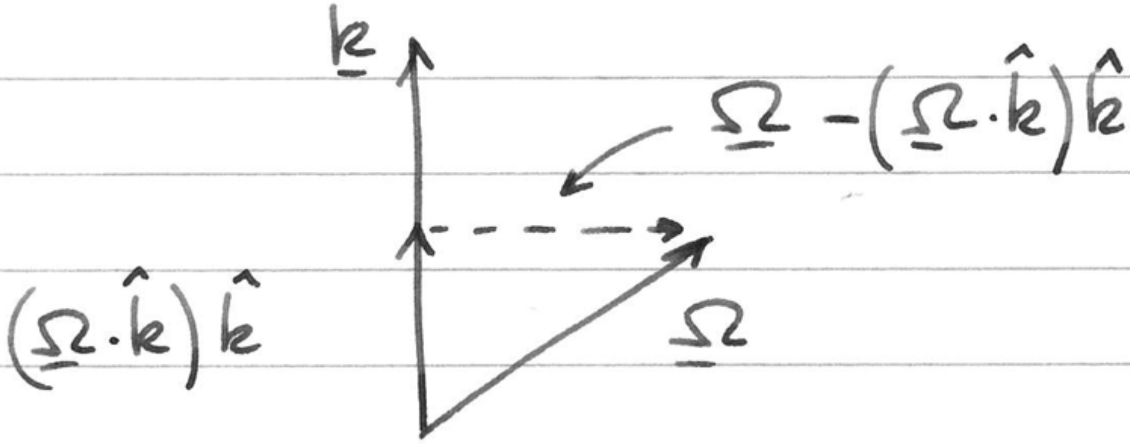
$$\Rightarrow \quad \sigma^2 k^2 - 4(\underline{\Omega}\cdot\underline{k})^2 = 0$$

$$\Rightarrow \quad \boxed{\sigma(k) = \pm 2\,\frac{\underline{\Omega}\cdot\underline{k}}{k}}$$ Dispersion relation

Phase velocity $\quad \underline{c}_\varphi = \dfrac{\sigma}{k^2}\underline{k} = \pm\, 2\dfrac{(\underline{\Omega}\cdot\underline{k})}{k^3}\underline{k}$ ← wave propagation

Group velocity:

$$\underline{c}_g = \underline{\nabla}_k \sigma = \pm\, 2\,\frac{k^2\underline{\Omega} - (\underline{\Omega}\cdot\underline{k})\underline{k}}{k^3}$$ ← energy propagation



Note that

$$\underline{c}_g \perp \underline{k} \parallel \underline{c}_\varphi$$

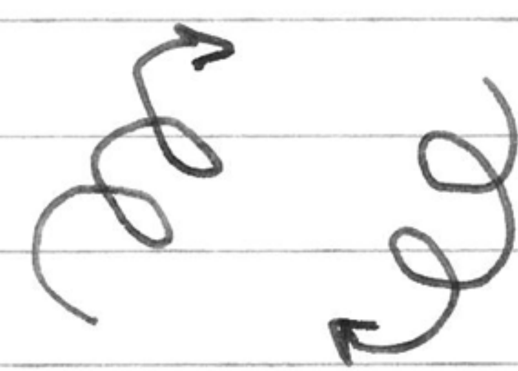
Besides $\quad \underline{\omega} = \nabla \times \underline{v} = i \underline{k} \times \underline{v}_0 \, e^{i(\underline{k}\cdot\underline{x} - \sigma t)}$

And $\quad \dfrac{\partial \underline{\omega}}{\partial t} = -i\sigma \underline{\omega} = i\, 2 (\underline{\Omega} \cdot \underline{k})\, \underline{v}$

$$\Longrightarrow \quad \underline{\omega} = \pm k \underline{v}$$

or $\quad i\hat{k} \times \underline{v}_0 = \pm \underline{v}_0$

and the waves are circularly polarized (helicoidal)



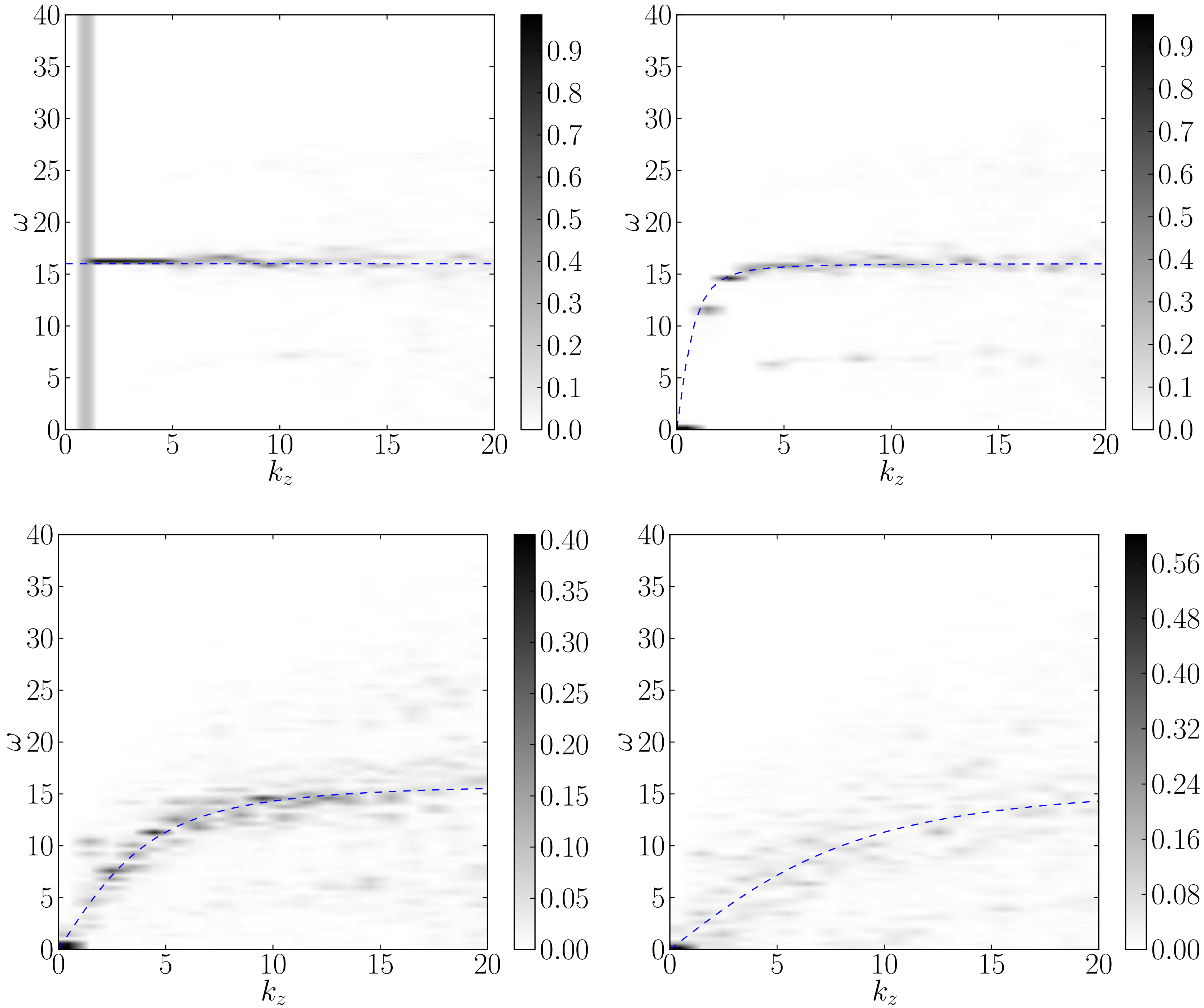
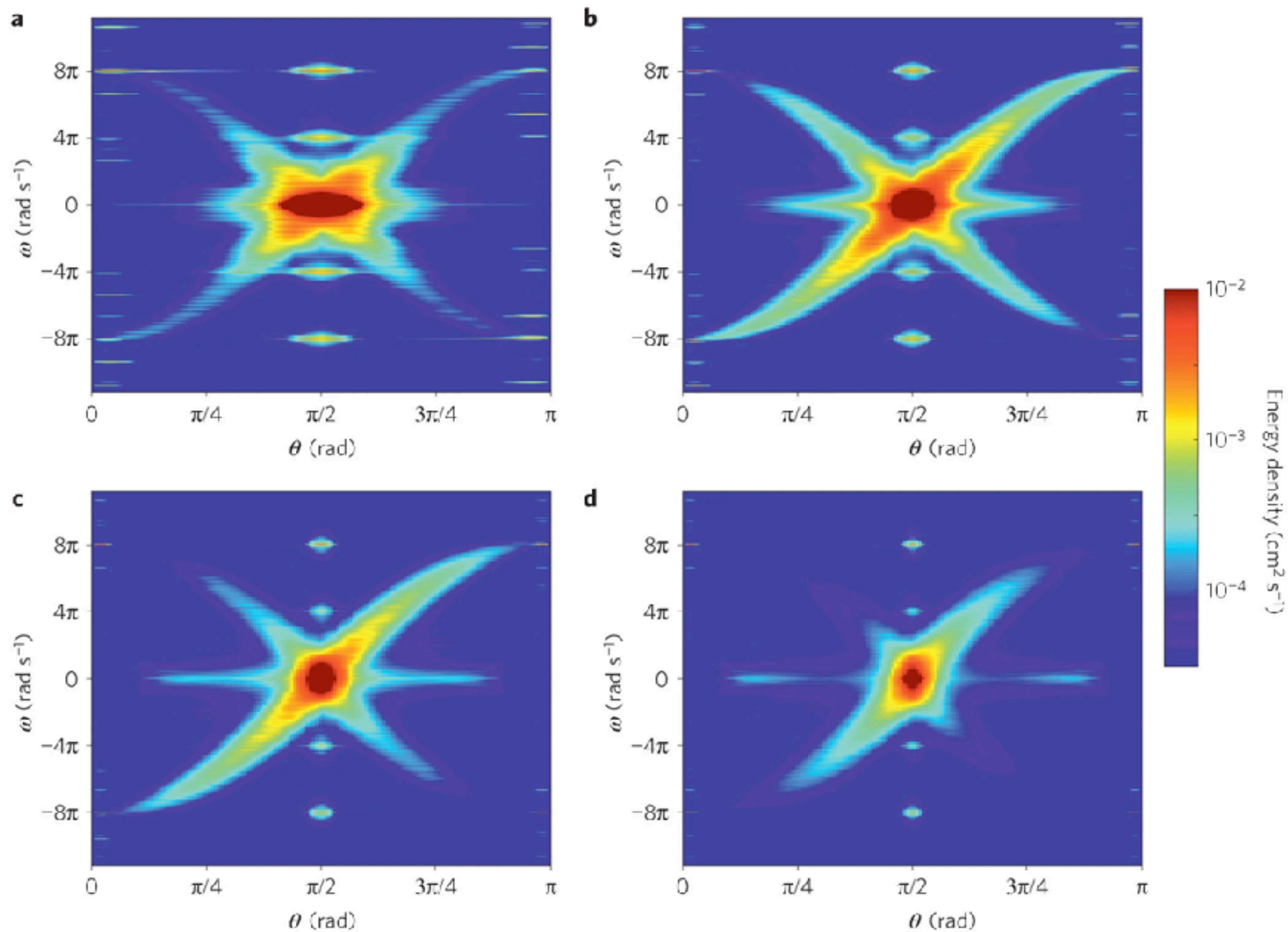


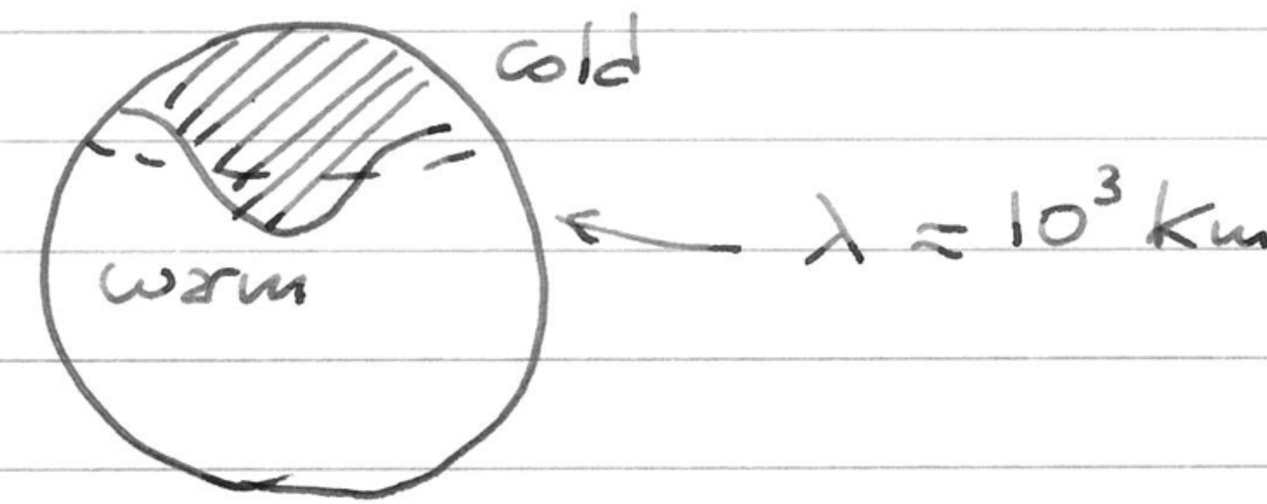
FIG. 3. Normalized wave vector and frequency spectrum $E_{11}(\mathbf{k}, \omega)/E_{11}(\mathbf{k})$ for the run with $\Omega = 8$. Darker regions indicate larger energy density. The dashed curve indicates the dispersion relation for inertial waves. (Top left) Normalized $E_{11}(k_x = 0, k_y = 0, k_z, \omega)$. (Top right) Normalized $E_{11}(k_x = 0, k_y = 1, k_z, \omega)$. (Bottom left) Normalized $E_{11}(k_x = 0, k_y = 5, k_z, \omega)$. (Bottom right) Normalized $E_{11}(k_x = 0, k_y = 10, k_z, \omega)$. Note from the maximum values in the color bars how the modes close to the dispersion relation concentrate most of the energy in the first two cases ($k_y = 0$ and $k_y = 1$), while as k_y is increased energy becomes more spread.

Clark Di Leoni et al., PoF (2014)



Yarom & Sharon, Nature Physics (2014)

If Ω depends on latitude we also have Rossby waves, which are slow traveling waves



These waves are also relevant in accretion disks.