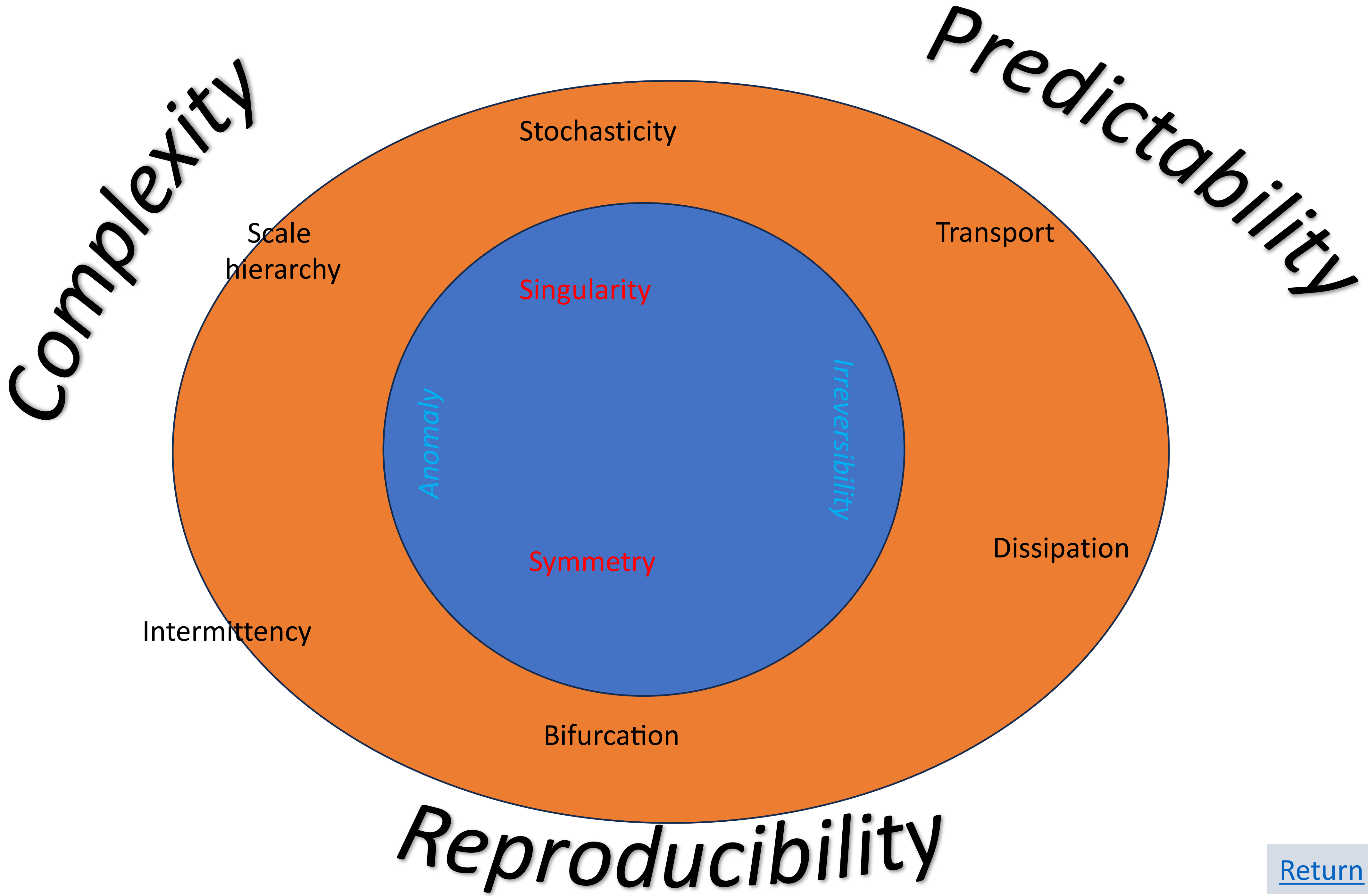
Class 2-B: Singularities

Physics of Turbulence

"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.

Sir Horace Lamb



Complexity
Predictability
Reproducibility
Stochasticity
Scale hierarchy
Transport
Singularity
Anomaly
Irreversibility
Symmetry
Dissipation
Intermittency
Bifurcation

Return

Sous chapter de intermittency (ou de dissipation?)
Parler de dissipative vs non dissipatives singularites)
Parler de leur recherche
-en log lattices(?)
-en experience
Parler de disparition de intermittence avec enlevement des DR

Some Mathematical aspects of Navier-Stokes equations

$$\vec{\nabla} \bullet \vec{u} = \vec{0}$$

$$\partial_t \vec{u} + (\vec{u} \bullet \vec{\nabla}) \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \nu \Delta \vec{u}$$

Symmetries

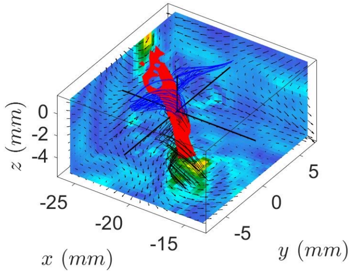
Tme-translation	$t \to t + h$
Space translation	$\vec{x} \to \vec{x} + \vec{h}$
Space-reversal	$(\vec{x}, \vec{u}) \to (-\vec{x}, -\vec{u})$
Galilean invariance	$(\vec{x}, \vec{u}) \to (\vec{x} + \vec{U}t, \vec{u} + \vec{U})$
Scaling	$(t, \vec{x}, \vec{u}) \to (\lambda^2 t, \lambda \vec{x}, \lambda^{-1} \vec{u}) \quad \nu \neq 0$

Model of singularity: homogeneous solution of NS of degree -1

Recaling Symmetry for h=-1 $(t, x, u) \longrightarrow (\gamma^2\, t, \gamma x, \gamma^{-1} u)(\,\nu \neq 0)$

$u(\gamma^2\, t, \gamma x) = \gamma^{-1} u(t, x)$ **homogeneous solutions of NS of degree -1**

Axisymmetric case: Landau in 1944 and further discussed by Batchelor



$$
\begin{aligned}
\nabla \cdot \mathbf{U} &= 0, \\
(\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{\nabla p}{\rho} - \nu \Delta \mathbf{U} &= \nu^2 \delta(\mathbf{x}) \mathbf{F},
\end{aligned} \tag{2.1}
$$

206 *Flow of a uniform incompressible viscous fluid* [4.6

symmetry about an axis, leaving r and θ (the angle subtended by the radius vector and the axis of symmetry) as the only independent variables, and proceed to impose more restrictions so that the dependence on either r or θ is made evident. The flow field to be discussed in this section may be obtained by postulating that the fluid velocity varies as r^{-1}, the dependence on θ then being given by an ordinary differential equation. This kind of procedure is indirect, in that we do not know what kind of flow field we have, or whether it is physically significant, until the mathematical solution has been interpreted, but it can be quite purposeful in experienced hands.

Li, Yan Yan, (2016)
H. Faller M2 Thesis
H. Faller et al, (2021)

Model of singularity: homogeneous solution of NS of degree -1

Stationary solutions of NSE with a force at the origin

$$
\begin{aligned}
\nabla \cdot \mathbf{U} &= 0, \\
(\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{\nabla p}{\rho} - \nu \Delta \mathbf{U} &= \nu^2 \delta(\mathbf{x}) \mathbf{F},
\end{aligned}
$$

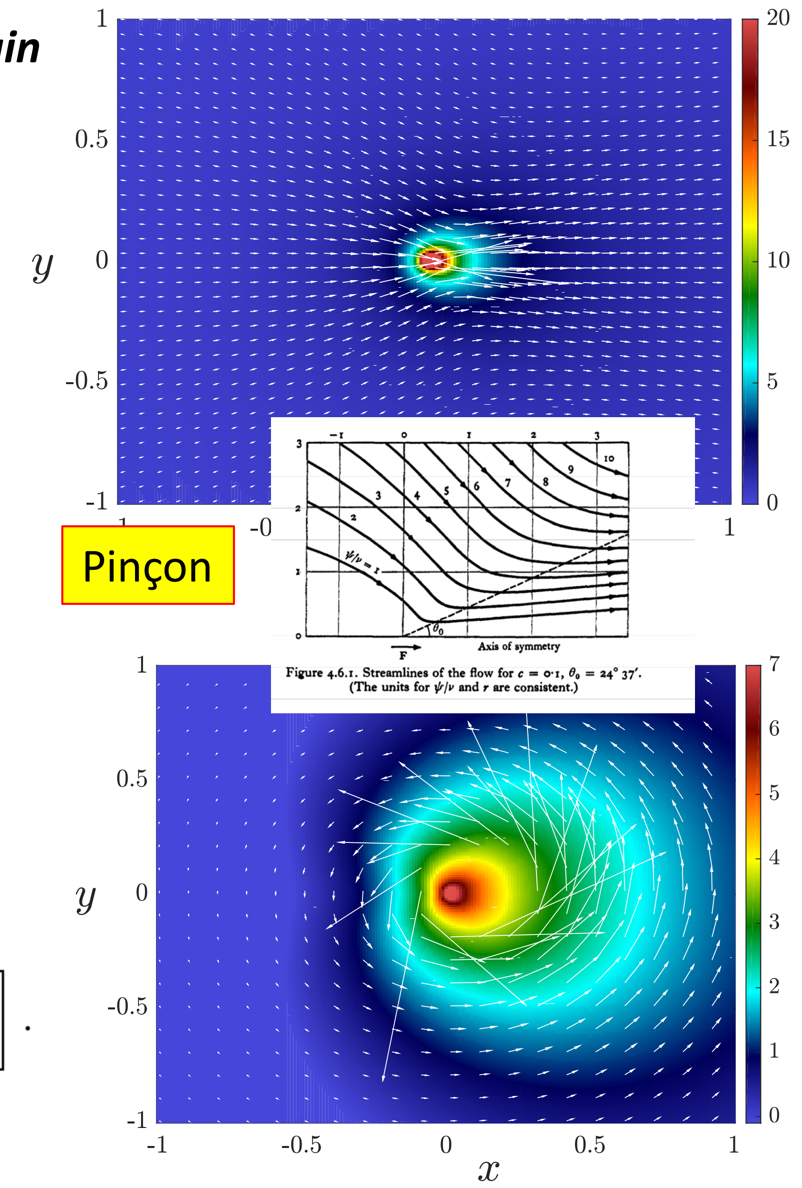
General form

$$
\phi(\mathbf{x}, \boldsymbol{\gamma}) = ||\mathbf{x}|| - \boldsymbol{\gamma} \cdot \mathbf{x}, \qquad \gamma < 1
$$

$$
\mathbf{U} = -2\nabla(\ln \phi) + 2\mathbf{x}\Delta \ln(\phi),
$$

and

$$
\begin{aligned}
\mathbf{F} &= F(||\boldsymbol{\gamma}||)\frac{\boldsymbol{\gamma}}{||\boldsymbol{\gamma}||}, \\
F(\gamma) &= 4\pi\left[\frac{4}{\gamma} - \frac{2}{\gamma^2}\ln\left(\frac{1+\gamma}{1-\gamma}\right) + \frac{16}{3}\frac{\gamma}{1-\gamma^2}\right].
\end{aligned}
$$



H. Faller et al submitted (2021)

Model of singularity: homogeneous solution of NS of degree -1

Stationary solutions of NSE with a force at the origin

$$\begin{aligned} \nabla \cdot \mathbf{U} &= 0, \\ (\mathbf{U}\cdot\nabla)\mathbf{U} + \frac{\nabla p}{\rho} - \nu \Delta \mathbf{U} &= \nu^2 \delta(\mathbf{x})\mathbf{F}, \end{aligned}$$

General form

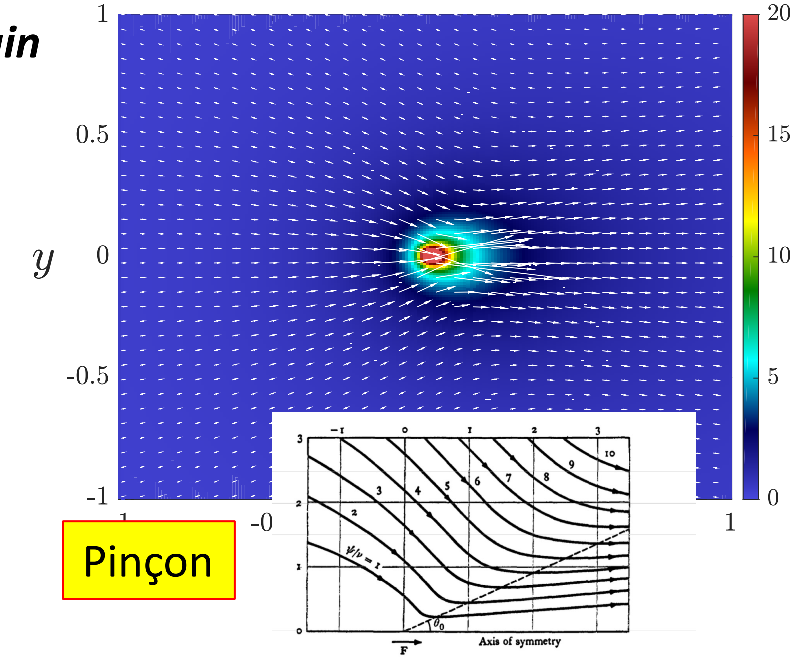
$$\phi(\mathbf{x}, \boldsymbol{\gamma}) = \|\mathbf{x}\| - \boldsymbol{\gamma}\cdot\mathbf{x}, \qquad \gamma < 1$$

$$\mathbf{U} = -2\nabla(\ln\phi) + 2\mathbf{x}\Delta\ln(\phi),$$

and

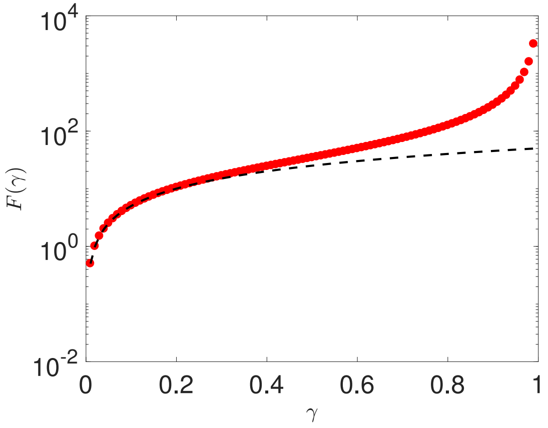
$$\mathbf{F} = F(||\boldsymbol{\gamma}||)\frac{\boldsymbol{\gamma}}{||\boldsymbol{\gamma}||},$$

$$F(\gamma) = 4\pi\left[\frac{4}{\gamma} - \frac{2}{\gamma^2}\ln\left(\frac{1+\gamma}{1-\gamma}\right) + \frac{16}{3}\frac{\gamma}{1-\gamma^2}\right].$$



Pinçon

Figure 4.6.1. Streamlines of the flow for $c = 0{\cdot}1$, $\theta_0 = 24°\,37'$. (The units for ψ/ν and r are consistent.)

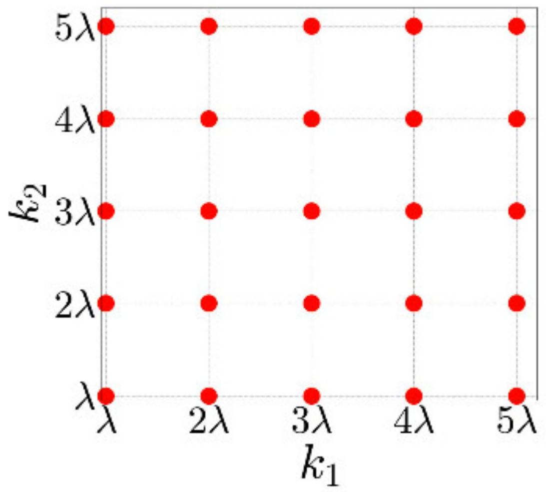


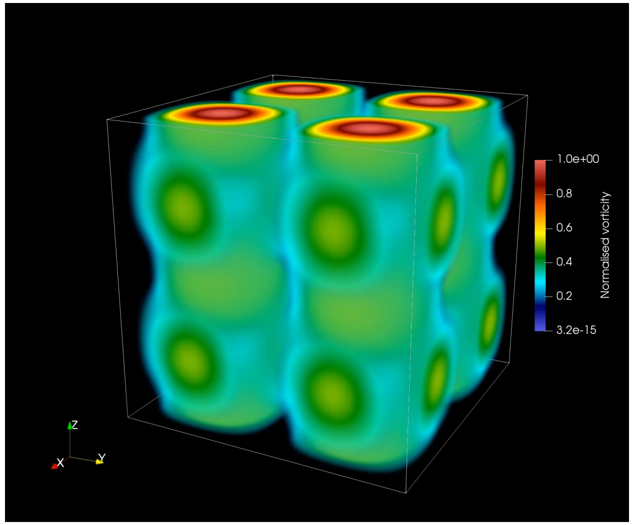
H. Faller et al submitted (2021)

Blow-up via DNS?

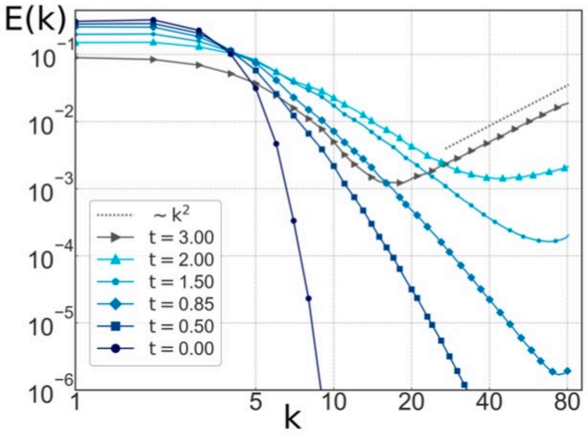
$$\partial_t \hat{u}_i + I k_j \hat{u}_j * \hat{u}_i = -I k_i \hat{P}$$

Fourier grid





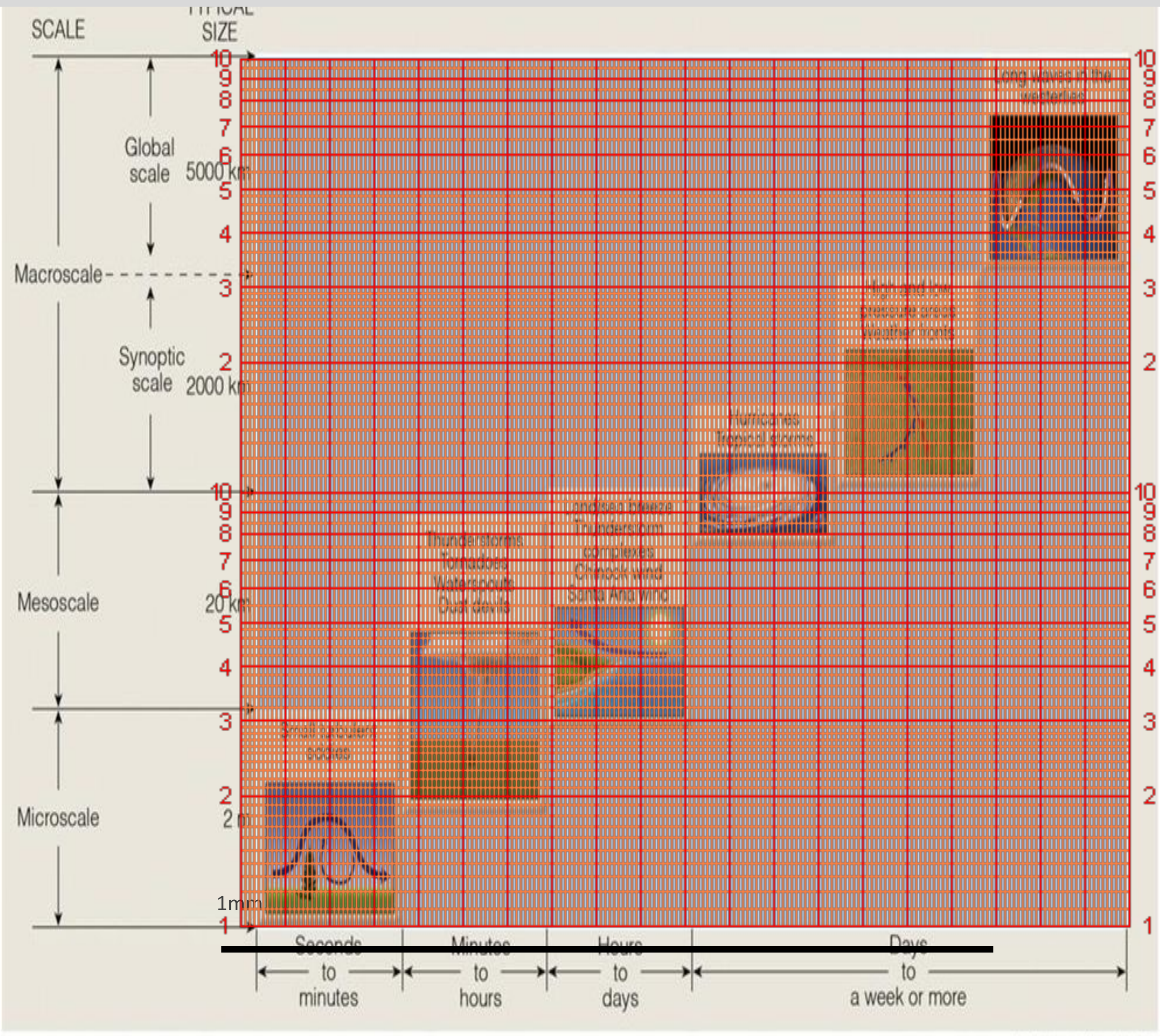
@J. Polanco



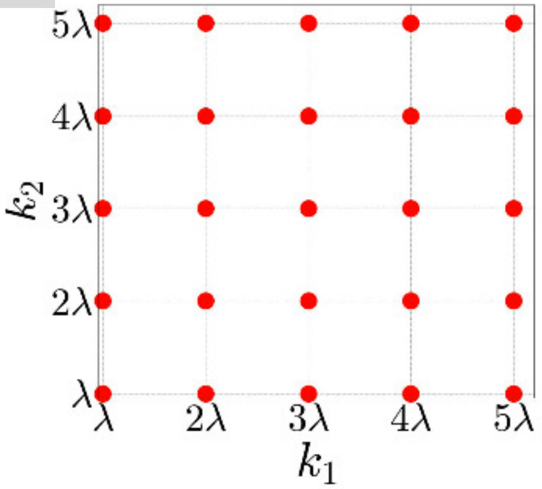
Conclusion:
We cannot follow a blow-up by DNS

Fluids on log lattices

From regular Fourier projection



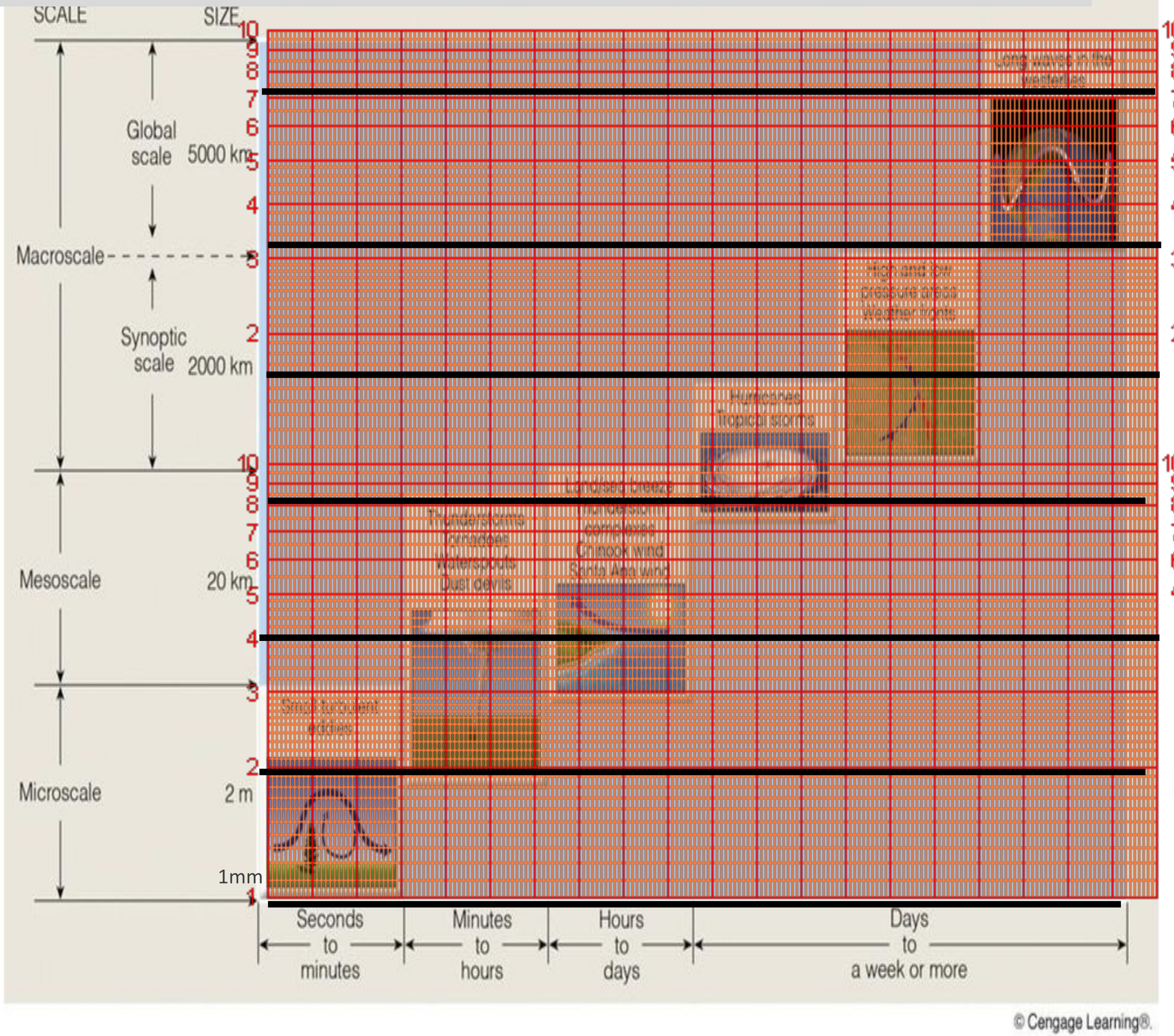
Fourier



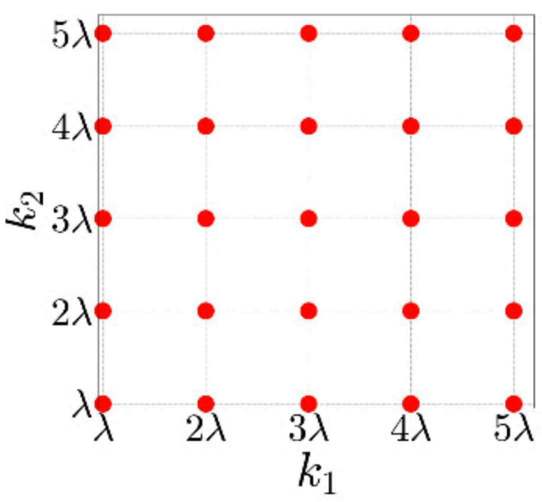
10^{24}

N^3 modes

To Fourier log-decimation

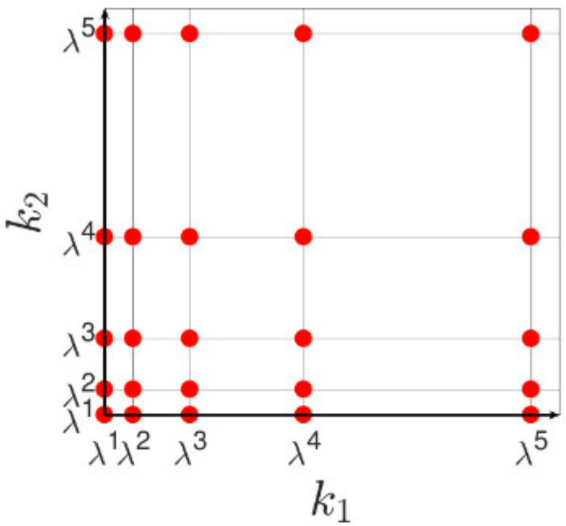


Fourier



10^{24}

N^3 modes



(Log(N))^3 modes

$6250=17^3$

Campolina&Mailybaev, 2018

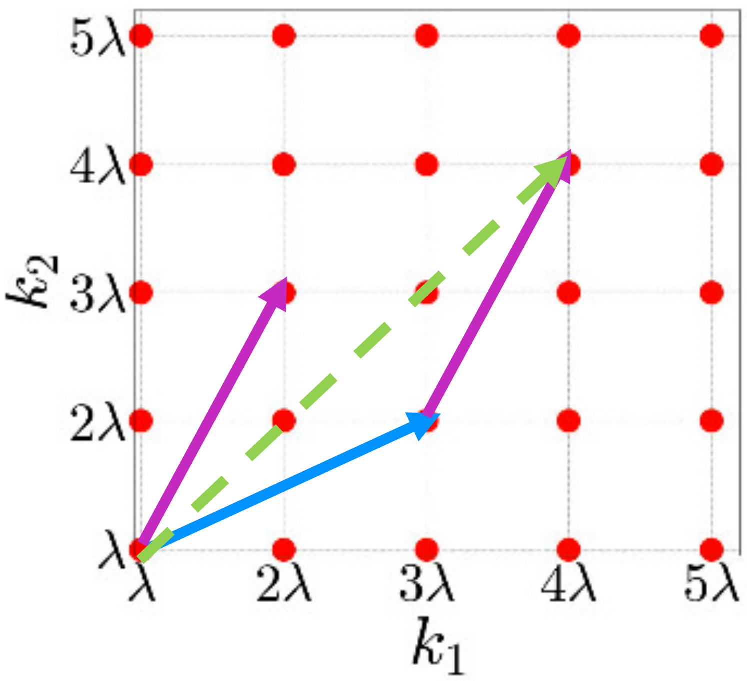
Avoiding aliasing

$$(f * g)(k) = \sum_{\boxed{p,q \in \Lambda}} c_{kpq} f(p) g(q)$$

p+q=k triads

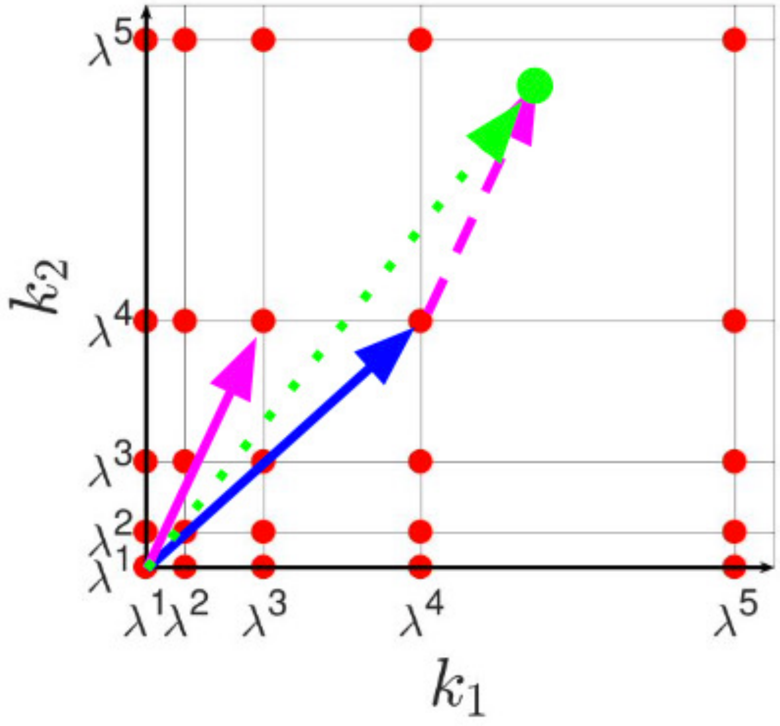
$$\partial_t \hat{u}_i = P_{ij} \left(-ik_q \hat{u}_q * \hat{u}_j + \hat{f}_j \right) - \nu_r k^2 \hat{u}_i,$$

Fourier grid



$m = n + q,\ (\mathrm{m,n,q}) \in \mathbb{Z}^3$

Log grid



$\lambda^m = \lambda^n + \lambda^q,\ (\mathrm{m,n,q}) \in \mathbb{Z}^3$

Campolina&Mailybaev, 2018

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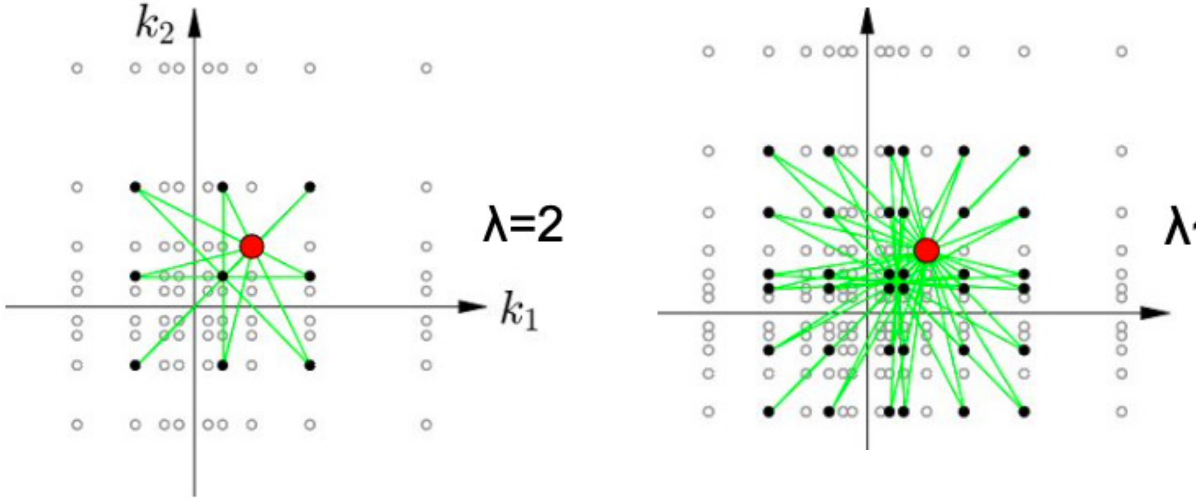
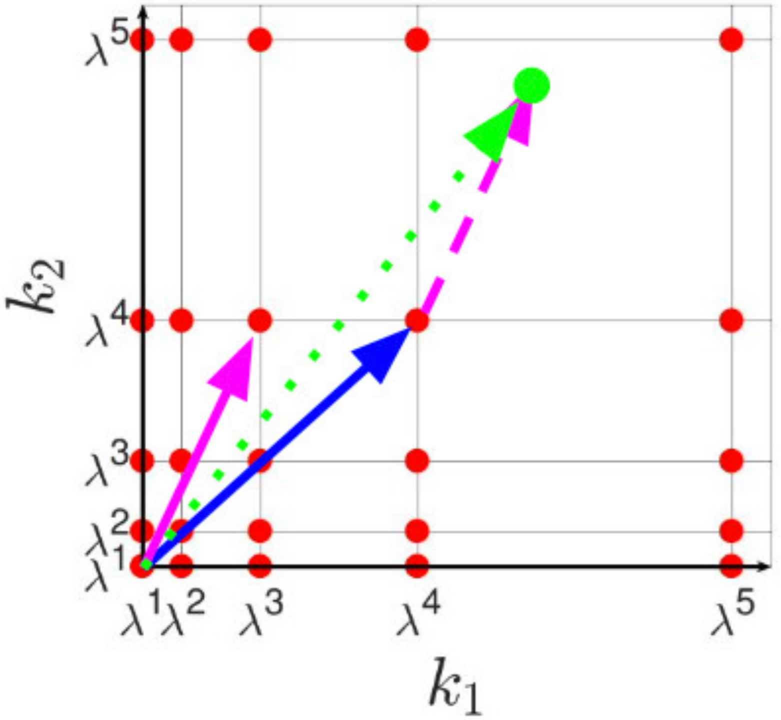
Log grid





$$\lambda^m = \lambda^n + \lambda^q,\ (\mathrm{m,n,q}) \in \mathbb{Z}^3$$

$$\lambda = 2 \quad (z = 3^D).$$
$$\lambda = \sigma \approx 1.325 \quad (z = 12^D)$$
$$\lambda = \Phi \approx 1.618 \quad (z = 6^D)$$
$$1 = \lambda^b - \lambda^a, 0 < a < b$$

Warning: log-lattice is not a mere 3D shell model: it is a true projection

Campolina&Mailybaev, 2018

To Fourier log-decimation

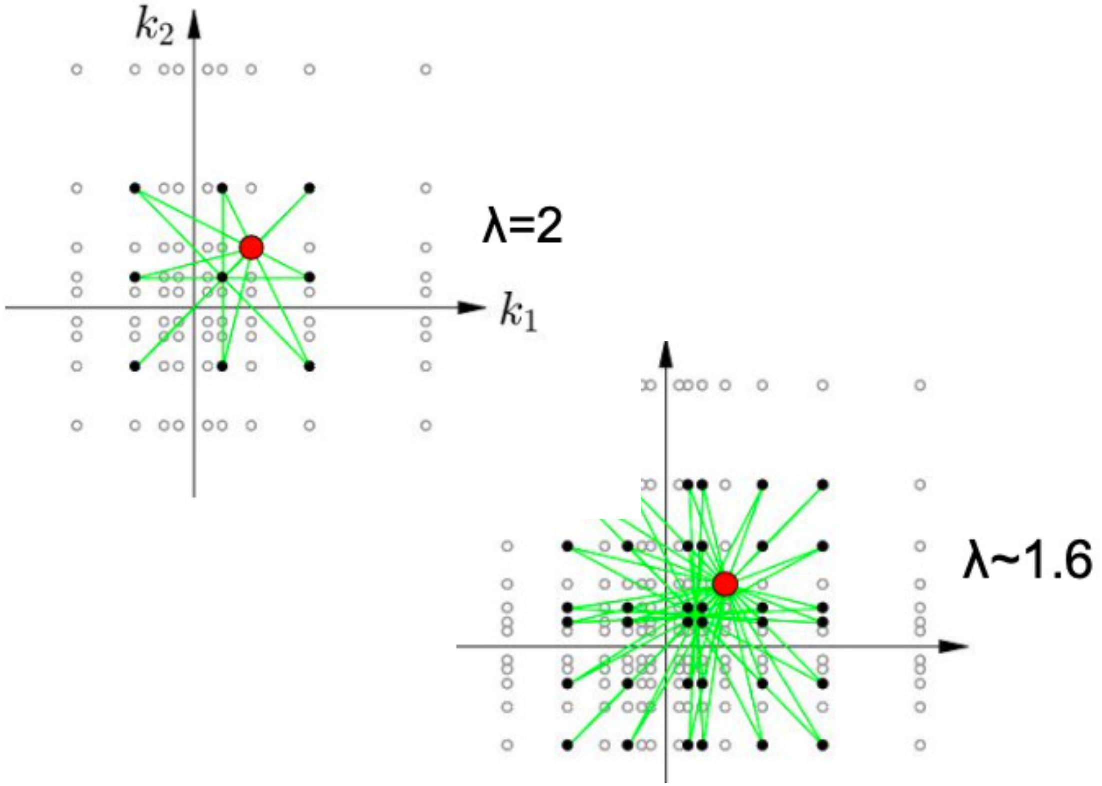
The projection on log-lattices is a true projection provided

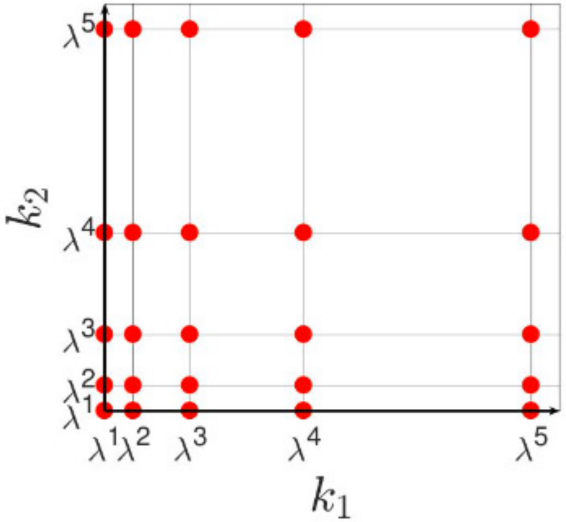
$$\lambda = 2 \quad (z = 3^D).$$
$$\lambda = \sigma \approx 1.325 \quad (z = 12^D)$$
$$\lambda = \Phi \approx 1.618 \quad (z = 6^D)$$
$$1 = \lambda^b - \lambda^a, 0 < a < b$$

The resulting fluid on log-lattices inherit all the symmetry and conservation laws of fluids

Kelvin theorem
Energy and Helicity in 3D
Energy and Enstrophy in 2D

Benefit: No adjustable parameter





$(\mathrm{Log}(N))^3$ modes

$6250=17^3$

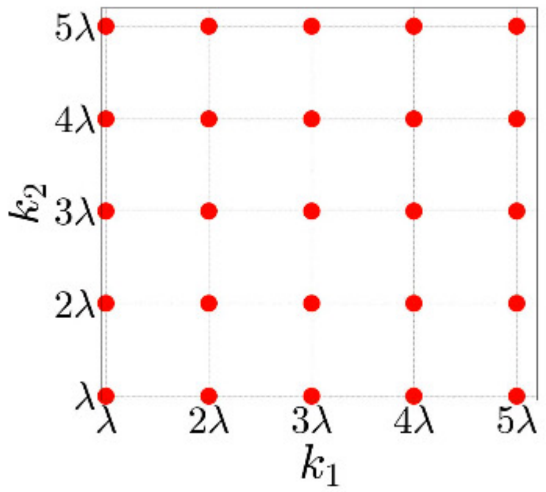
Campolina&Mailybaev, 2018

Singularities in fluid on log lattices

from DNS to log-lattices

$$\partial_t \hat{u}_i + I k_j \hat{u}_j * \hat{u}_i = -I k_i \hat{P}$$

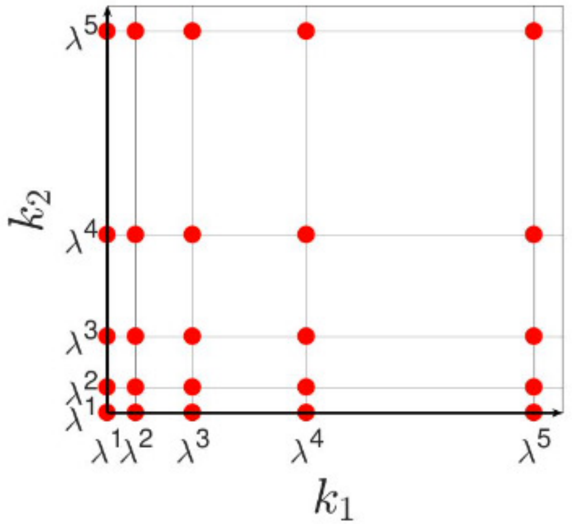
Fourier grid



N^3 modes

10^{24}

Log grid



$(\mathrm{Log}(N))^3$ modes

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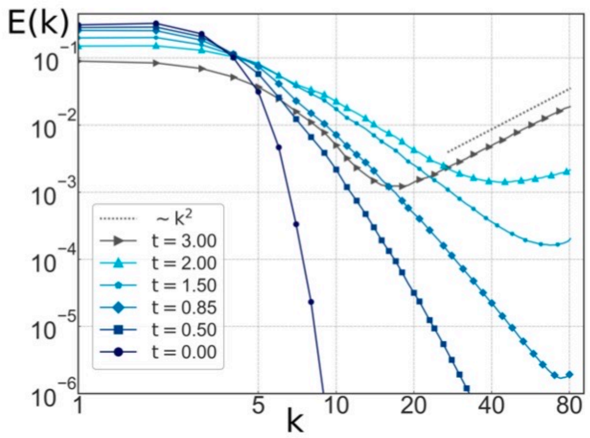
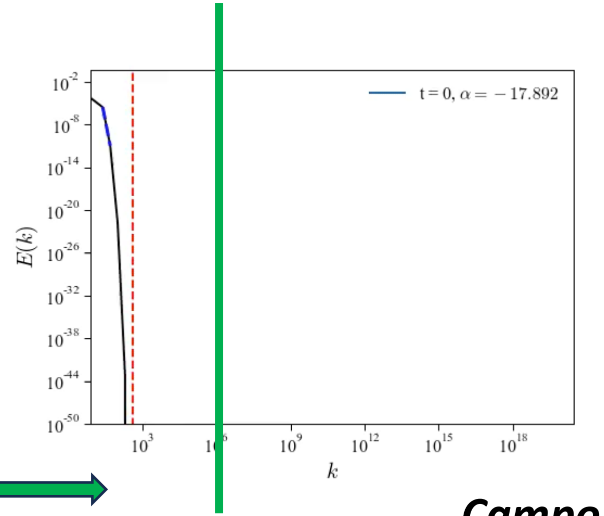




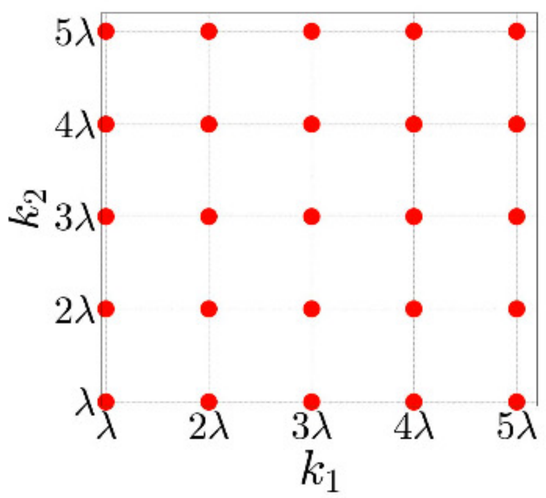
Max of existing DNS

Campolina&Mailybaev, 2018

from DNS to log-lattices

$$\partial_t \hat{u}_i + I k_j \hat{u}_j * \hat{u}_i = -I k_i \hat{P}$$

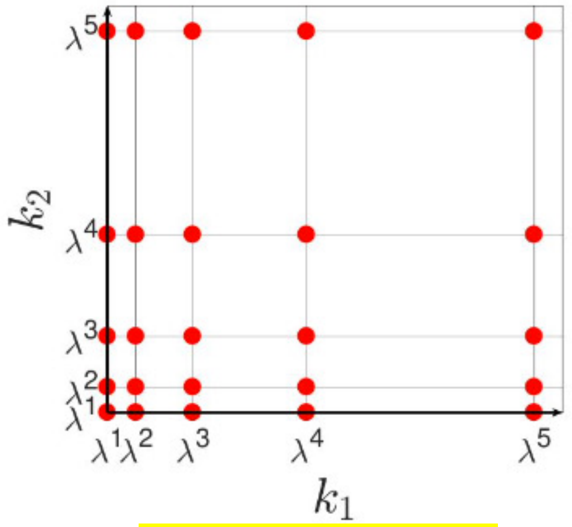
Fourier grid



N^3 modes

10^{24}

Log grid



$(\mathrm{Log}(N))^3$ modes

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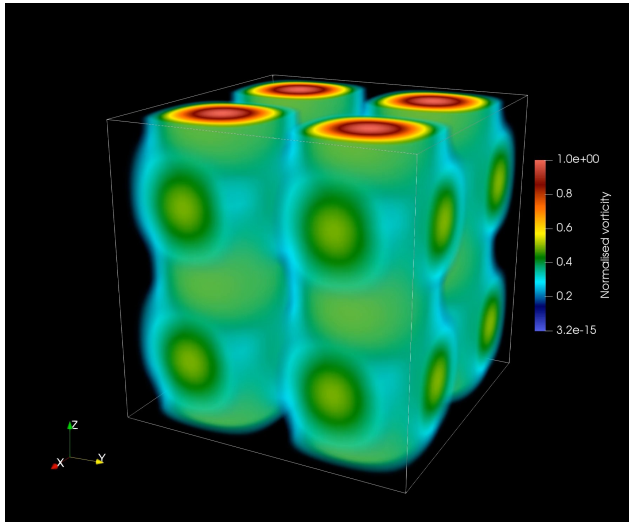
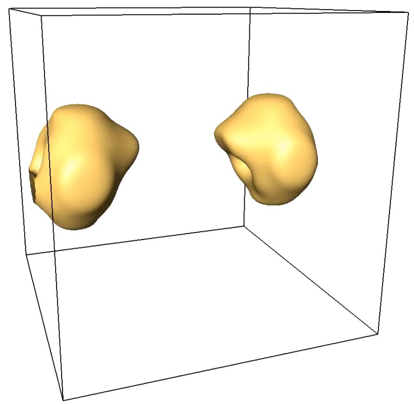


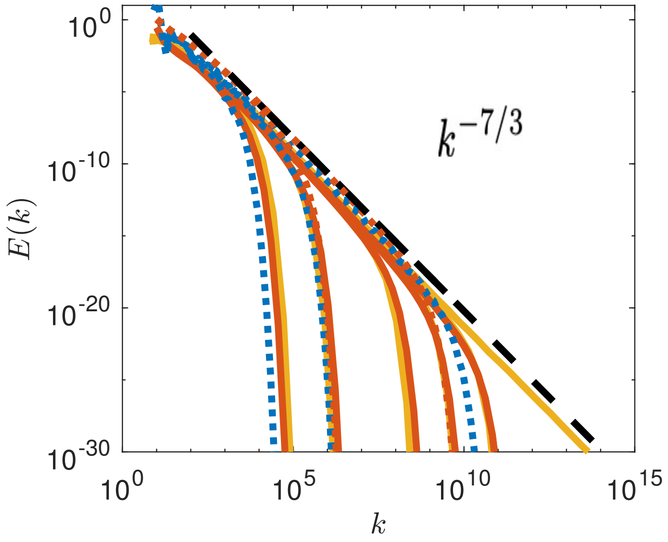
@J. Polanco

Vortex reconnection - Euler



@A. Harekrishnan

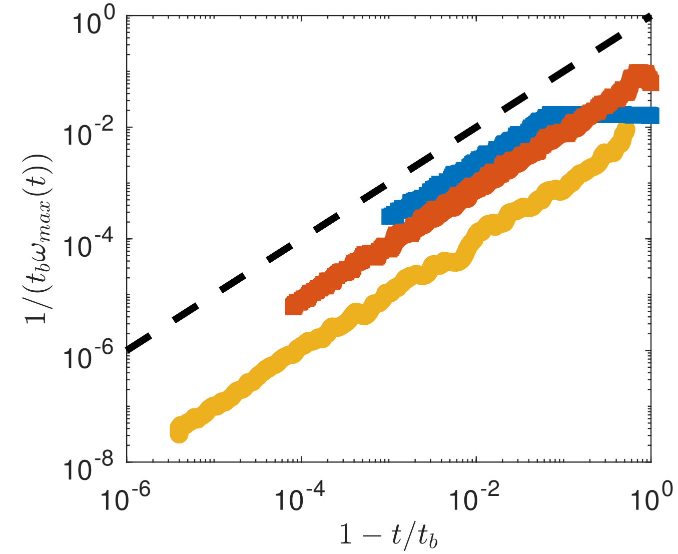
Blow-up in Euler 3D on log-lattice (adaptative grid)



$$\partial_t \hat{u}_i + I k_j \hat{u}_j * \hat{u}_i = -I k_i \hat{P}$$

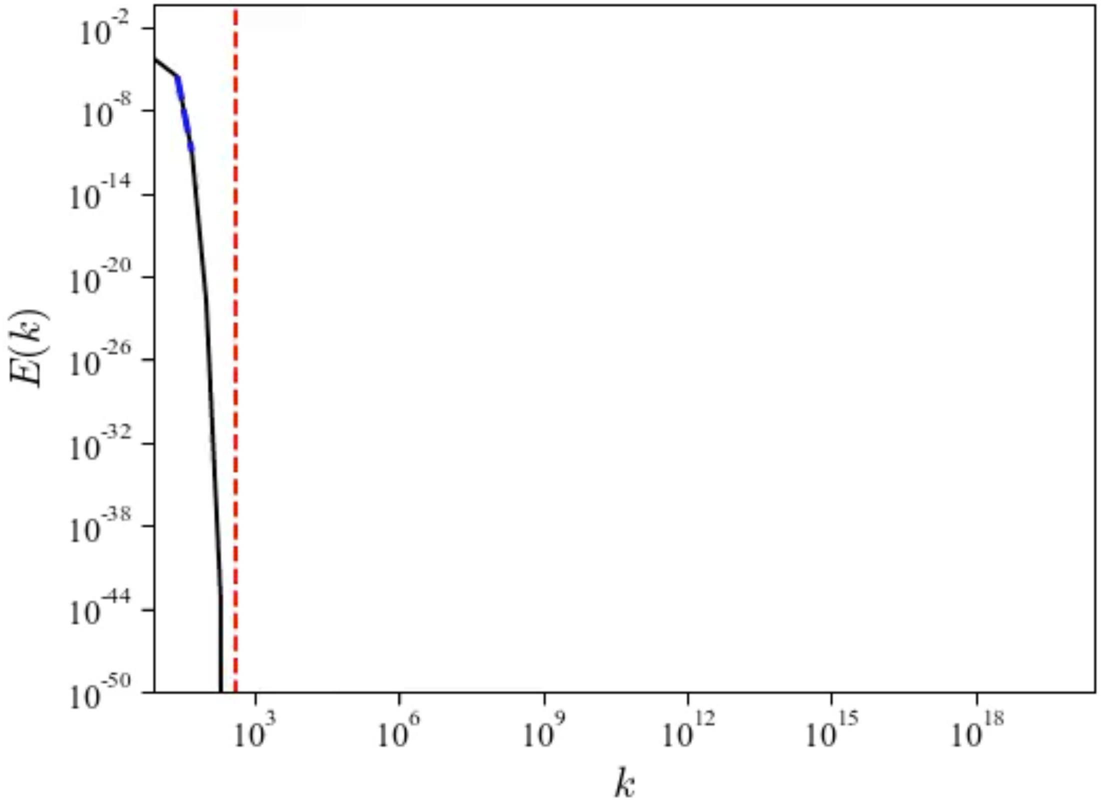
No viscosity: Energy is conserved

λ=2
λ=1.6
λ=1.3



Formation of a
Finite-time singularity

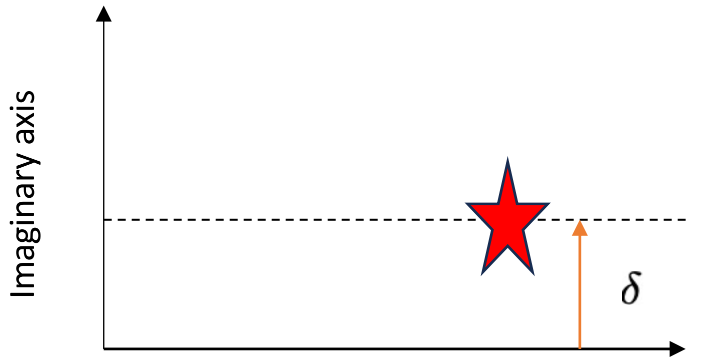
$$\omega \sim \frac{1}{t_b - t}$$



Campolina&Mailybaev, 2018

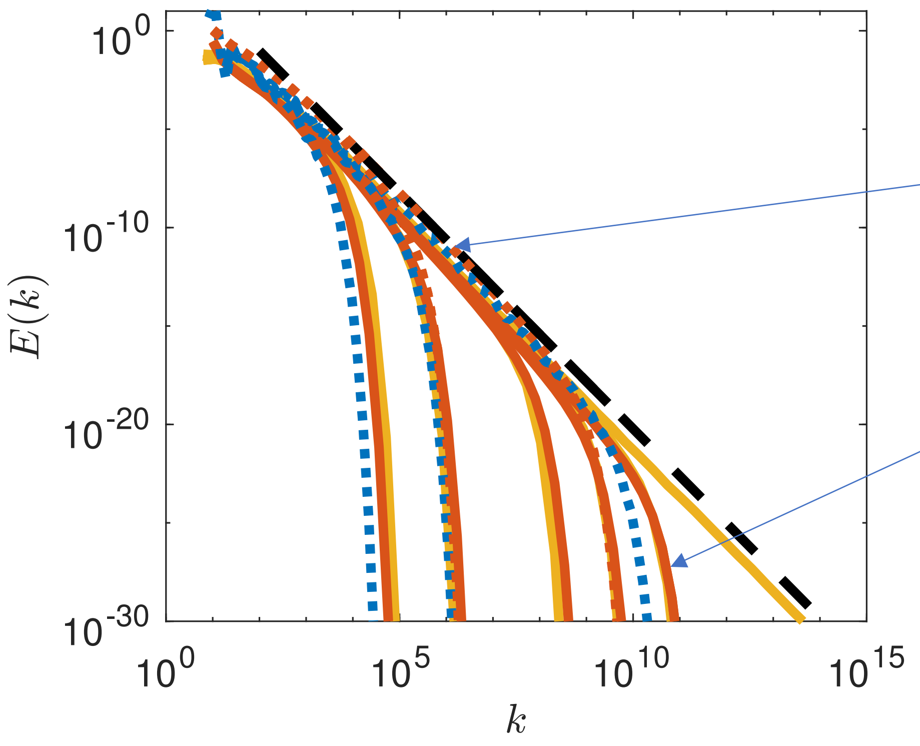
Singularities in complex plane

$$u(z) = (z - z_* - I\delta)^\alpha$$
$$\alpha < 1$$

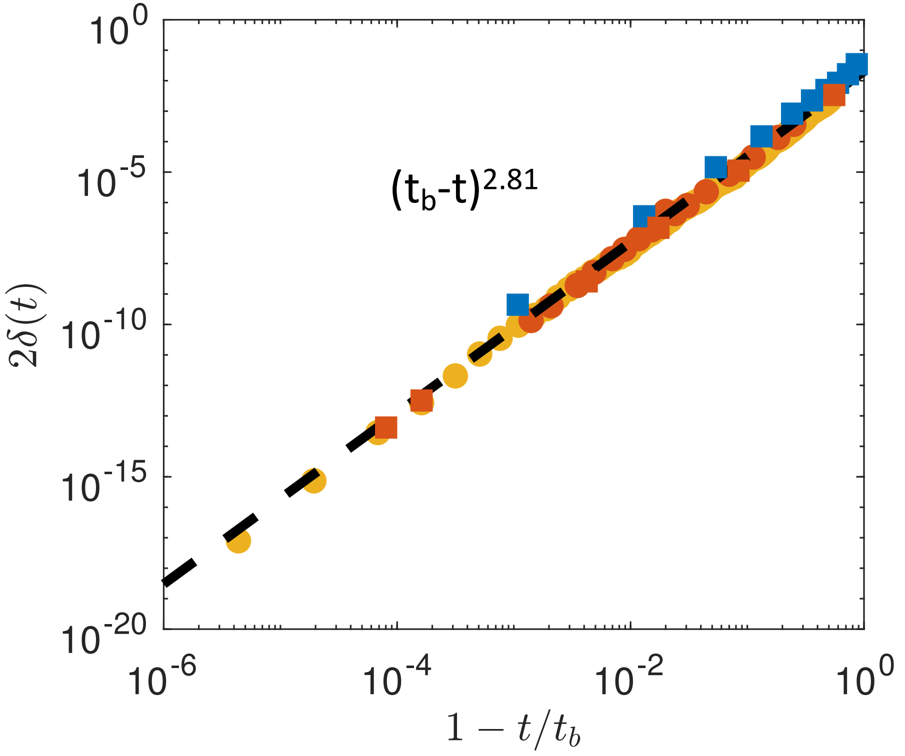


$$E(k) \sim k^{-2\alpha-1} \sin(kz_*) \exp(-2\delta k)$$

Frisch&Morf, 1981



Computed using the singularity strip method

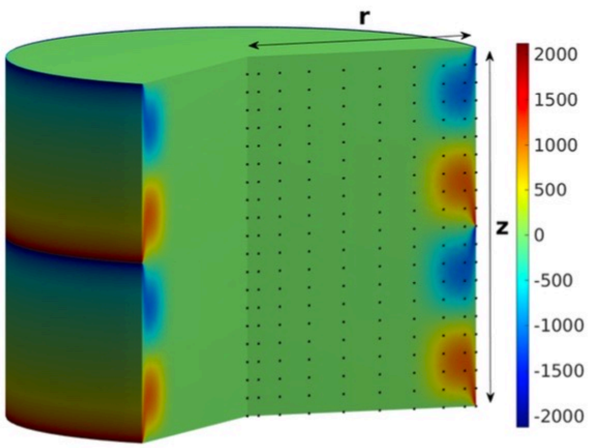


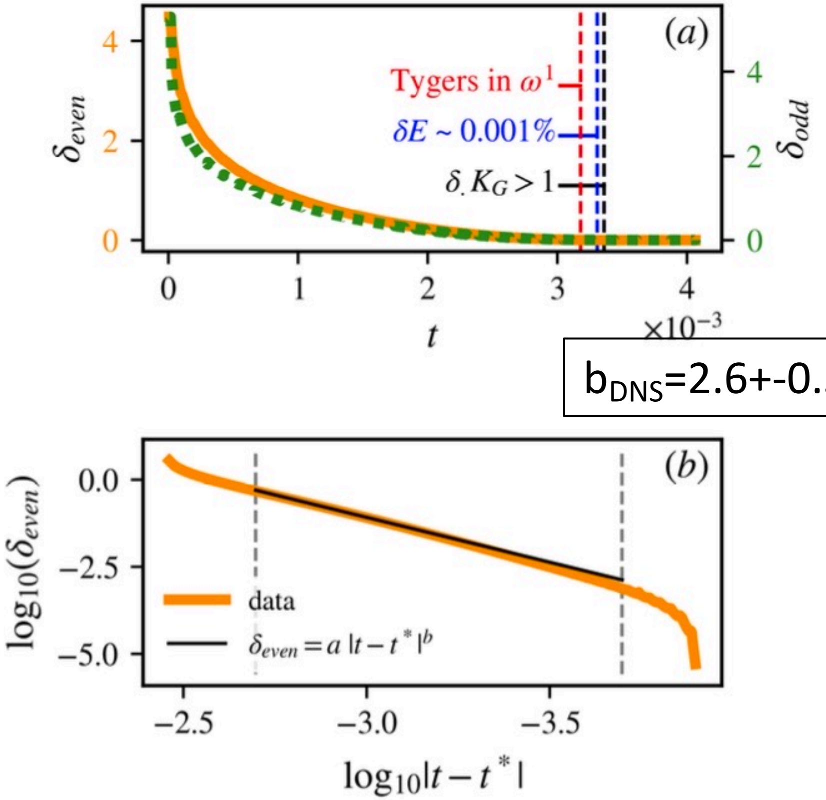
Pikeroen et al, 2023

Case of axisymmetric Euler

$$\partial_t u + u \cdot \nabla u = -\nabla p$$

Numerical solution





b_{DNS}=2.6+-0.5

Kolluru&Pandit, 2024

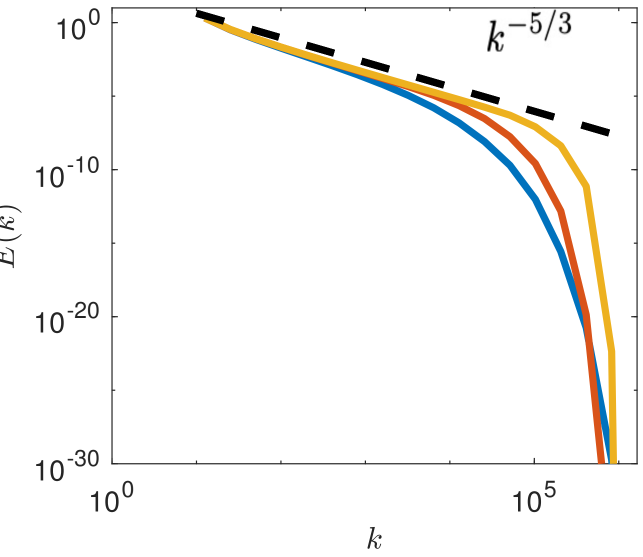
Same exponent than for 3D log-lattice

b_{LL}=2.81

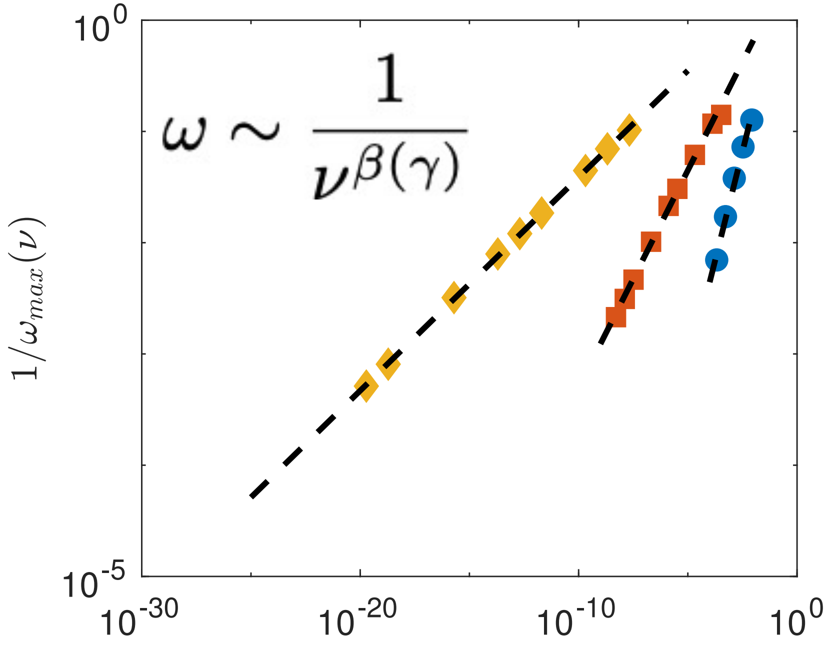
Influence of viscosity: $\gamma>1/3$

$$\partial_t \hat{u}_i = P_{ij}\left(-ik_q\hat{u}_q * \hat{u}_j + \hat{f}_j\right) \boxed{-\nu\, k^{2\gamma}\, u_i}$$

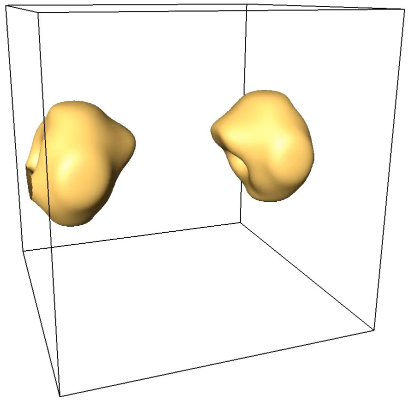
$$\nu\Delta u \rightarrow \quad \Delta^{\gamma} u$$



γ=2
γ=1
γ=1/2

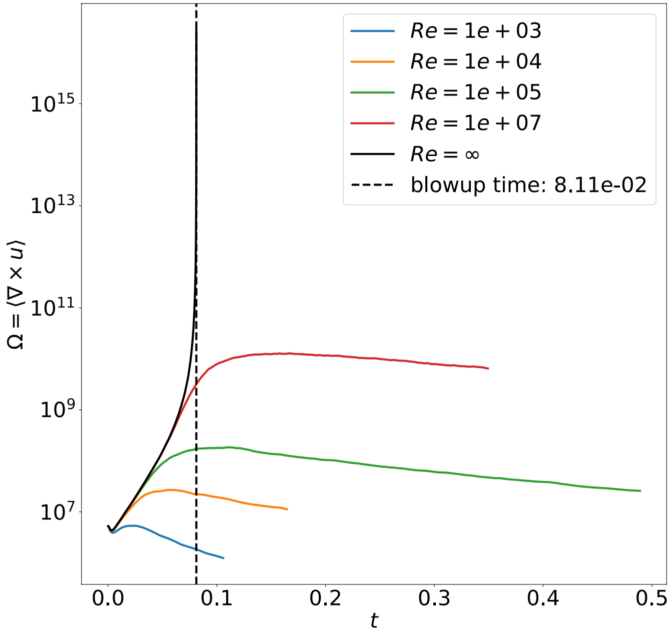


Vortex reconnection - Euler



No Blow-up in finite time

Possible singularities when $\nu \rightarrow 0$



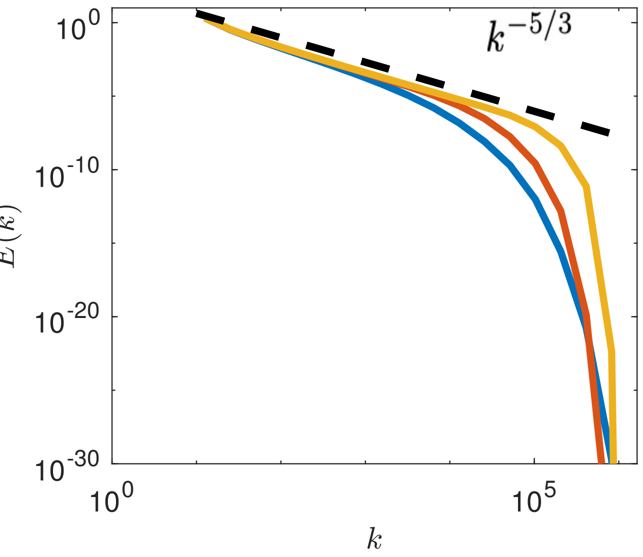
NB: for milder dissipation, finite time blow-up!

Pikeroen et al, 2023

Influence of viscosity: $\gamma>1/3$

$$\partial_t \hat{u}_i = P_{ij}\left(-ik_q\hat{u}_q * \hat{u}_j + \hat{f}_j\right) \quad -\nu\, k^{2\gamma} u_i$$

$$\nu\Delta u \rightarrow \quad \Delta^\gamma u$$



γ=2
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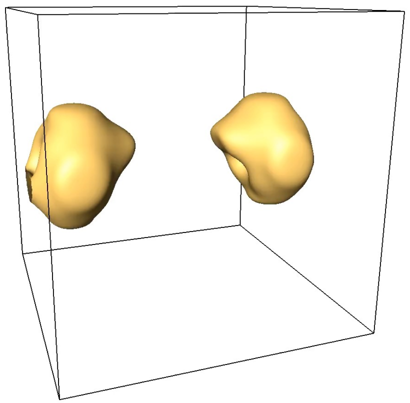
Vortex reconnection - Euler

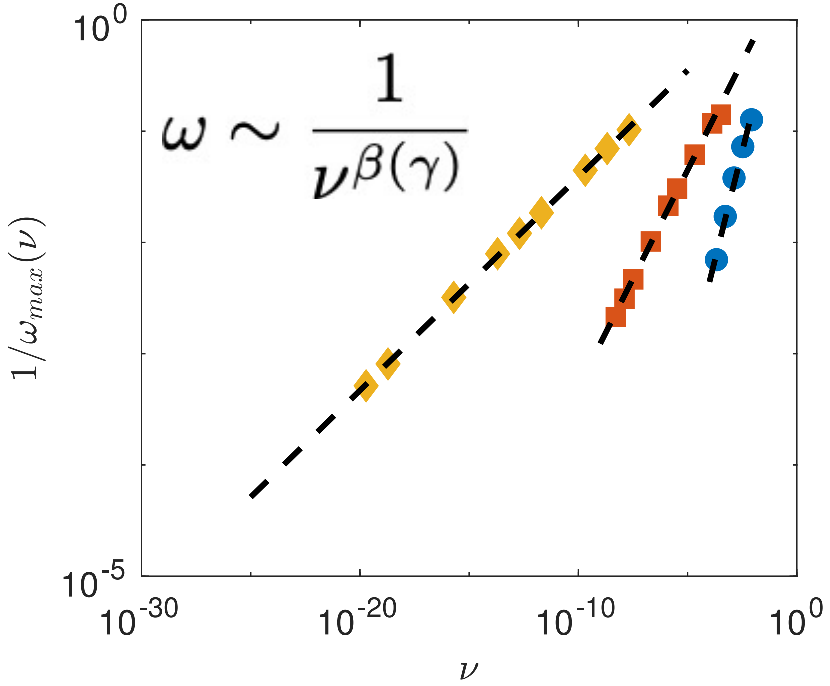


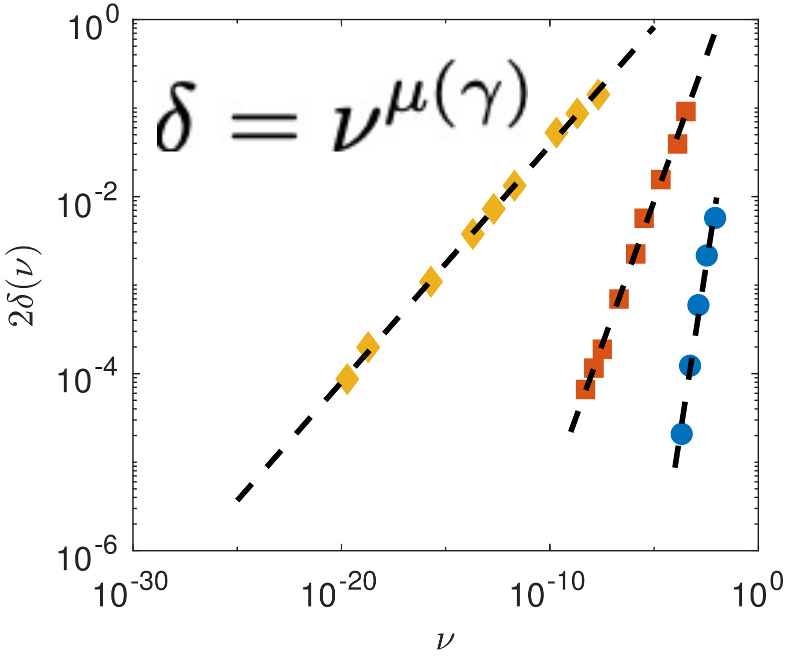
Pikeroen et al, 2023

No Blow-up in finite time

Possible singularities when $\nu \rightarrow 0$

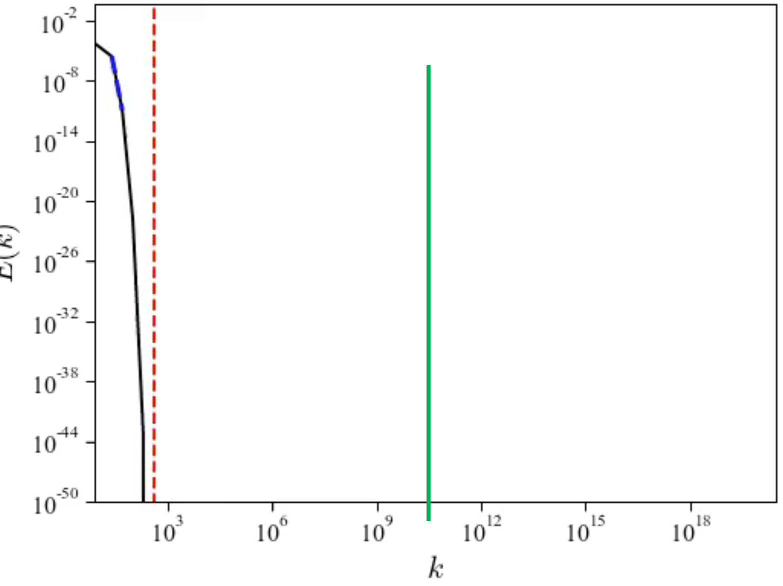
NB: for milder dissipation, finite time blow-up!



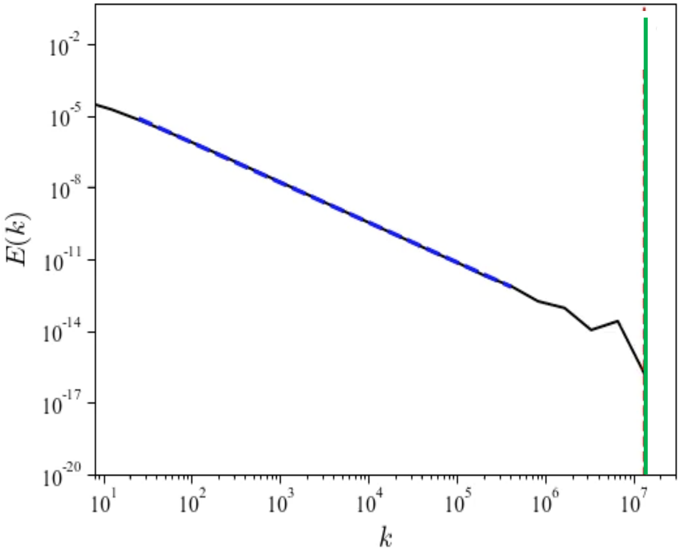


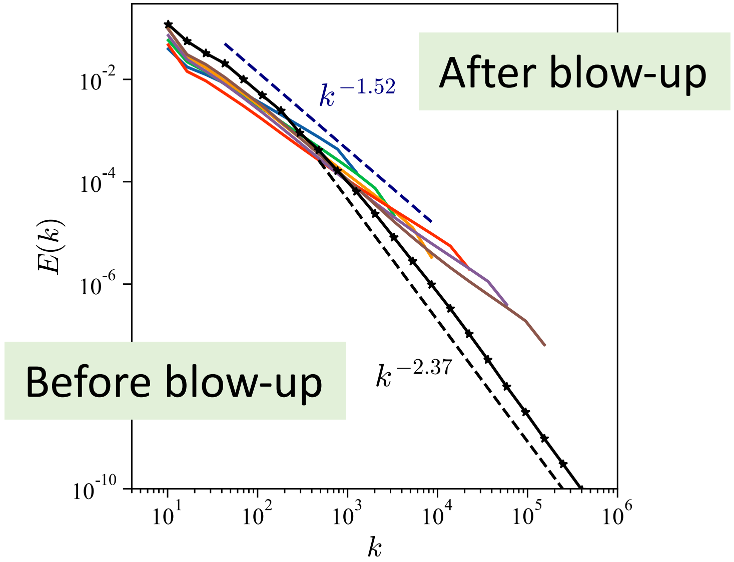
After the blow-up: convergency to another solution (dissipative)

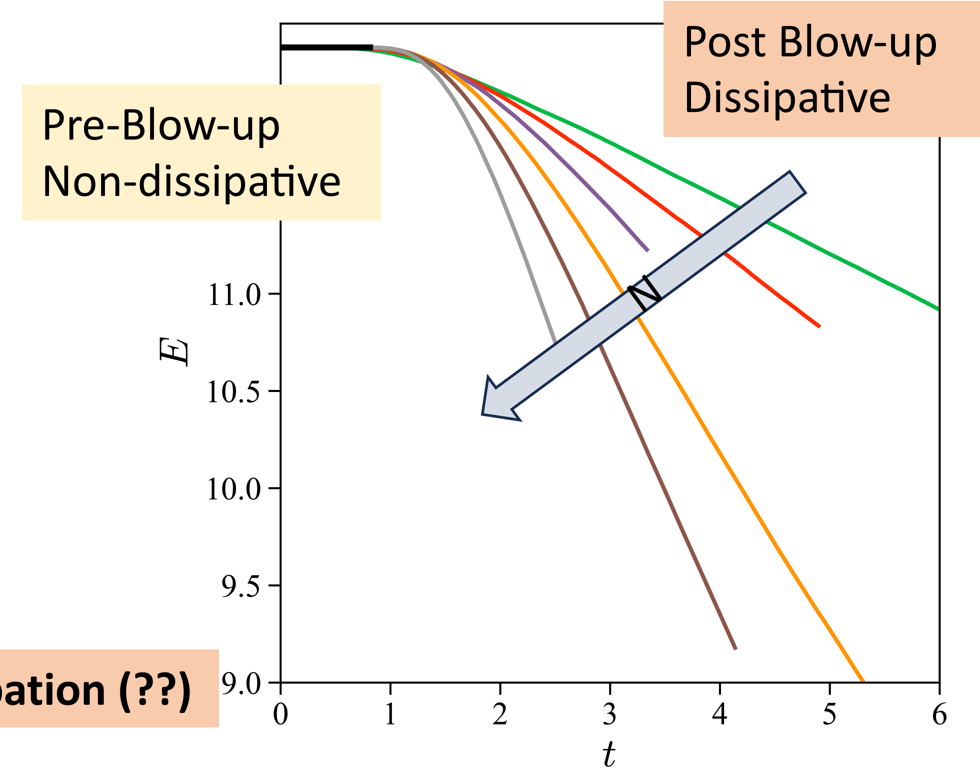
Before blow-ip



After blow-up







Regularization: **: impose a boundary in scale space and add a noise of decreasing intensity**

We fix the maximal wavenumber $k = \lambda^N$

$$dW = U(\lambda^N)d\eta; \quad < d\eta^2 >= 1; \quad < d\eta = 0 >$$

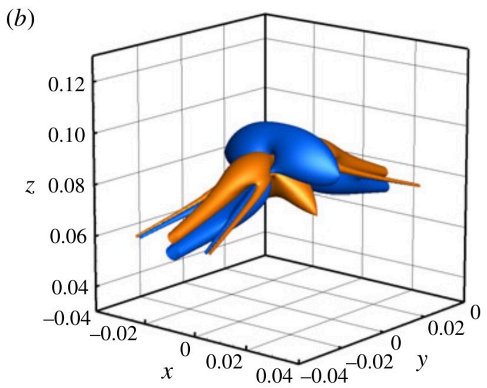
We look at limit $N \to \infty$

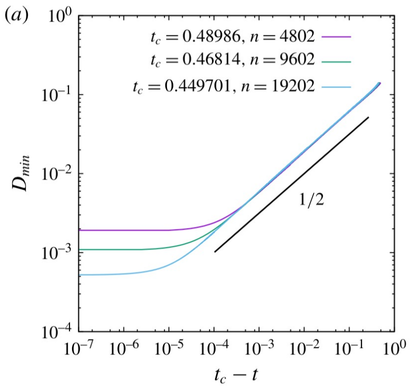
Singularity induces dissipation (??)

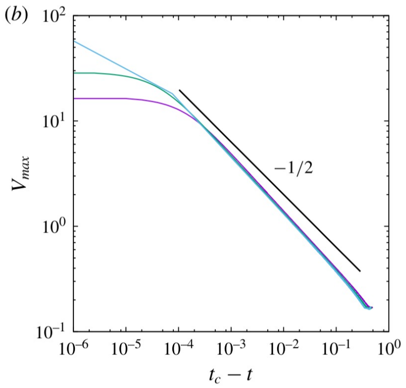
Costa et al, 2024

Mecanisms of singularity?

Reconnection and Blow-up



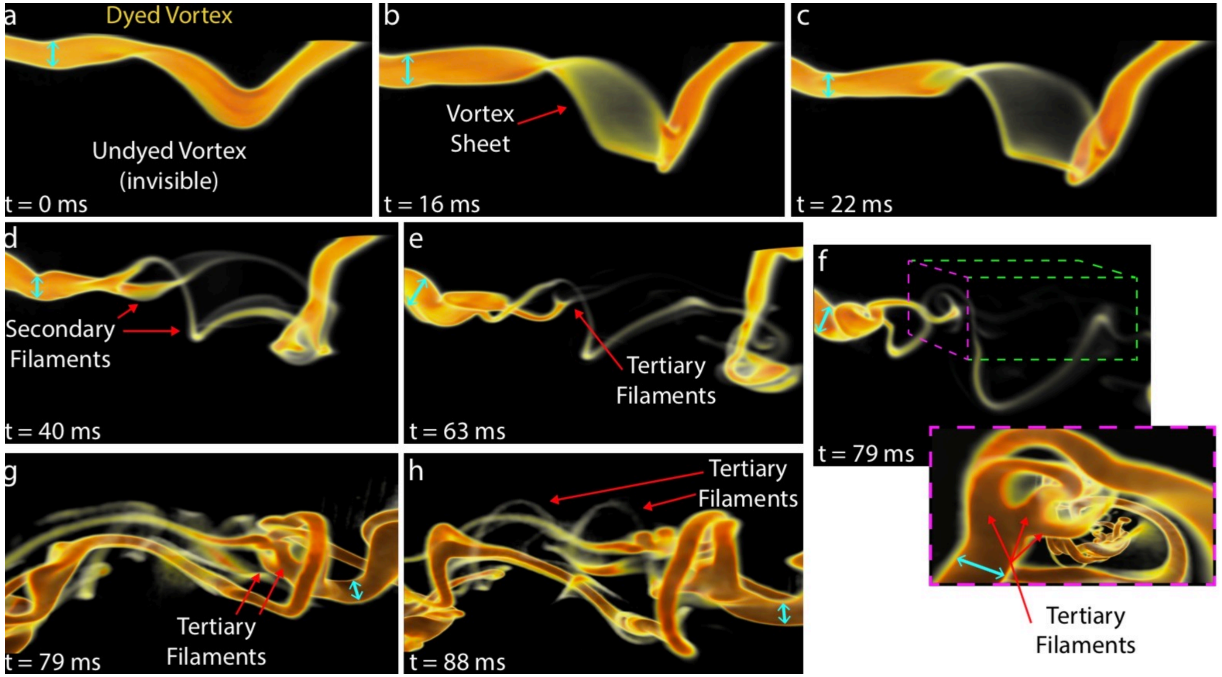




Biot-Savart self-similar evolution
Blow-up?

Kimura&Moffatt JFM (2017), (2020)

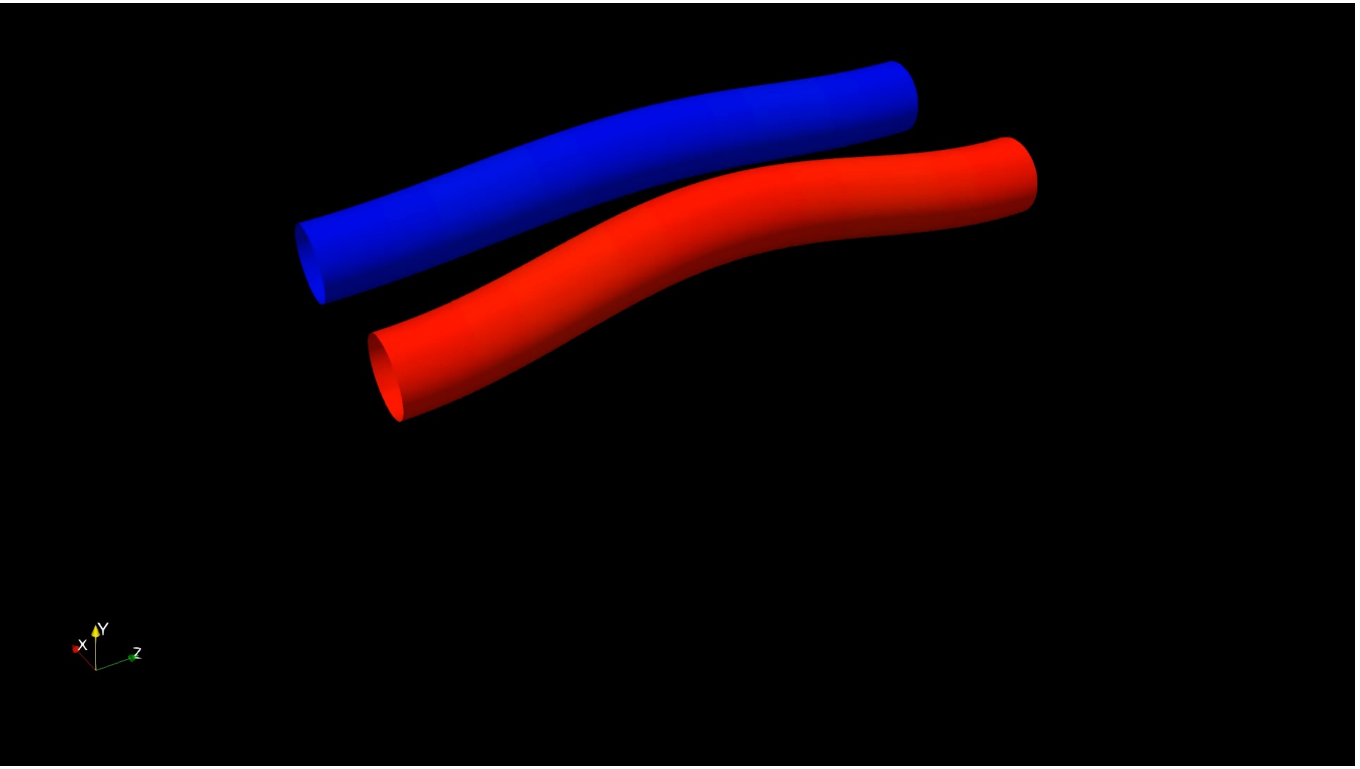
Interaction of vorticity structure:rings

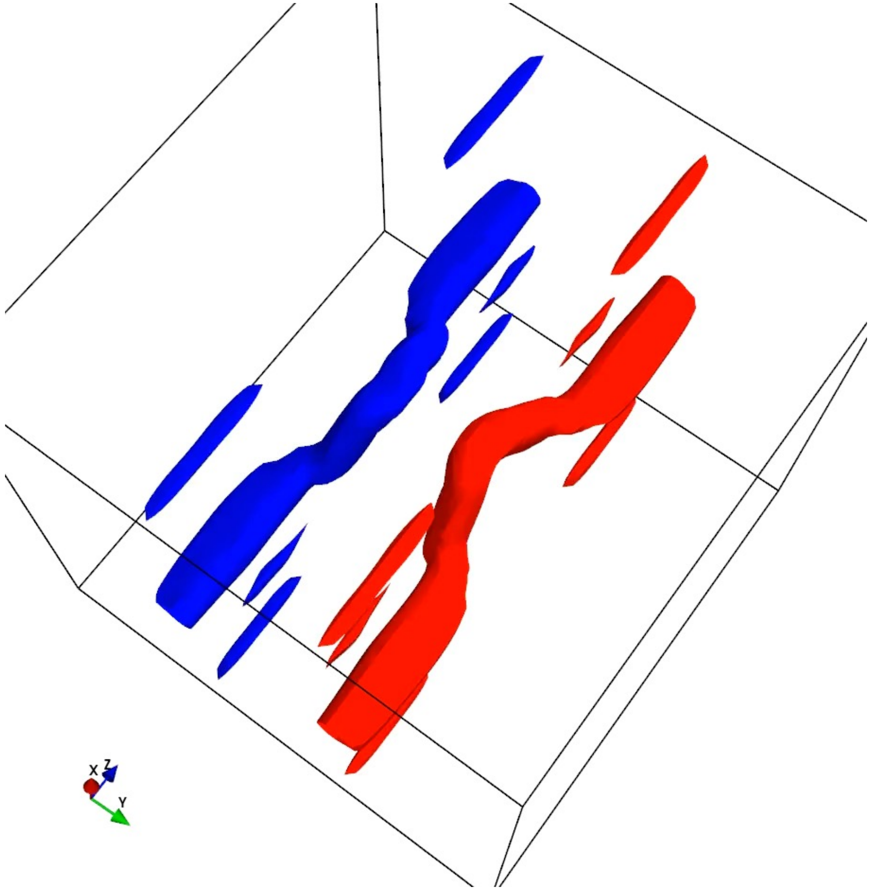


Lim&Nickels, Nature (1992); ,Mc Keown et al Science Advances (2020)

Interaction of vorticity structures: filaments

DNS





Log-Lattice

Harikrishnan et al JFM (2025)

Yao & Hussain JFM (2020)