Once over the sea,
winter winds can no longer
return home again
(Seishi Yamaguchi)

Lagrangian Turbulence: from tracers to intermittency and transport (III)

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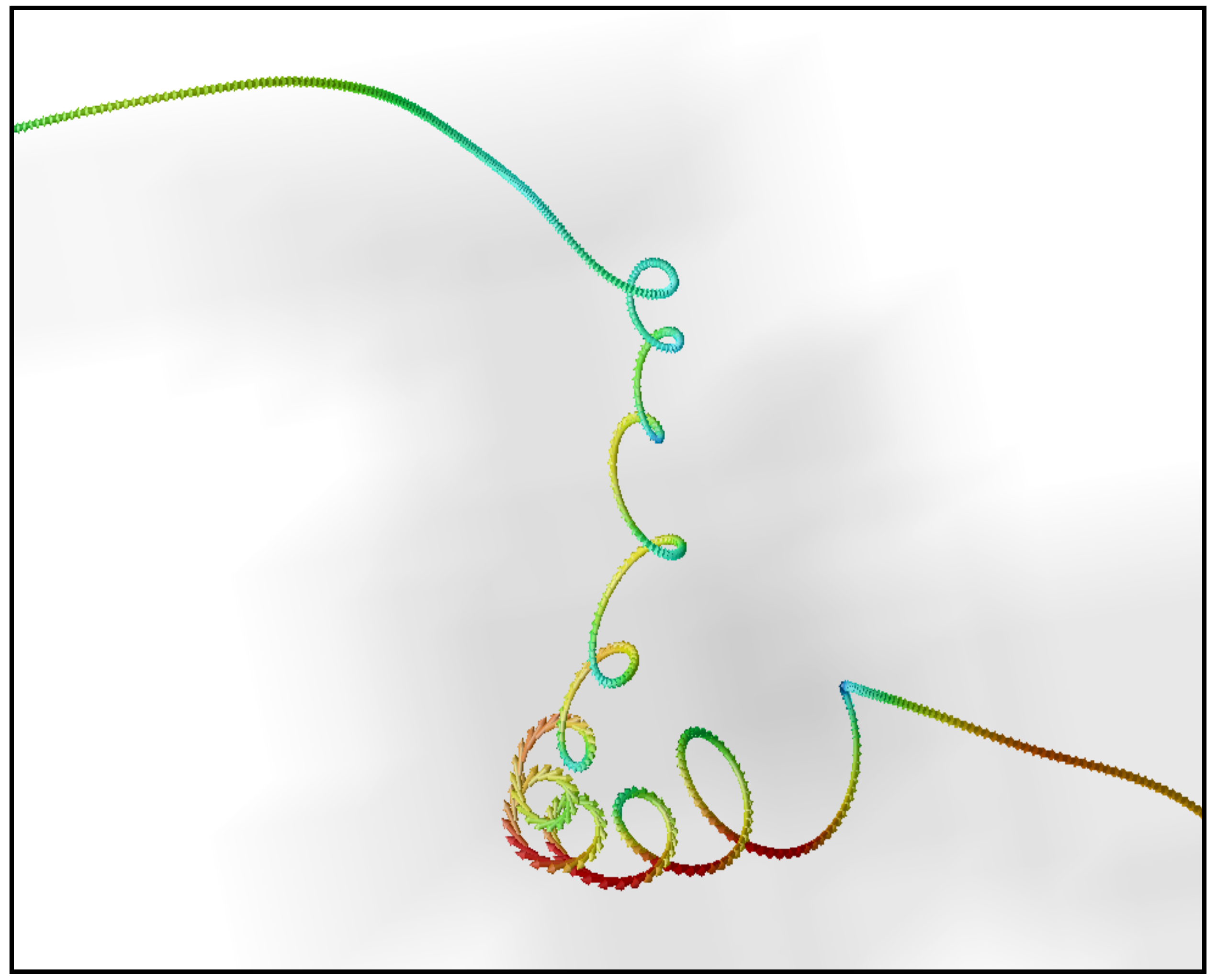
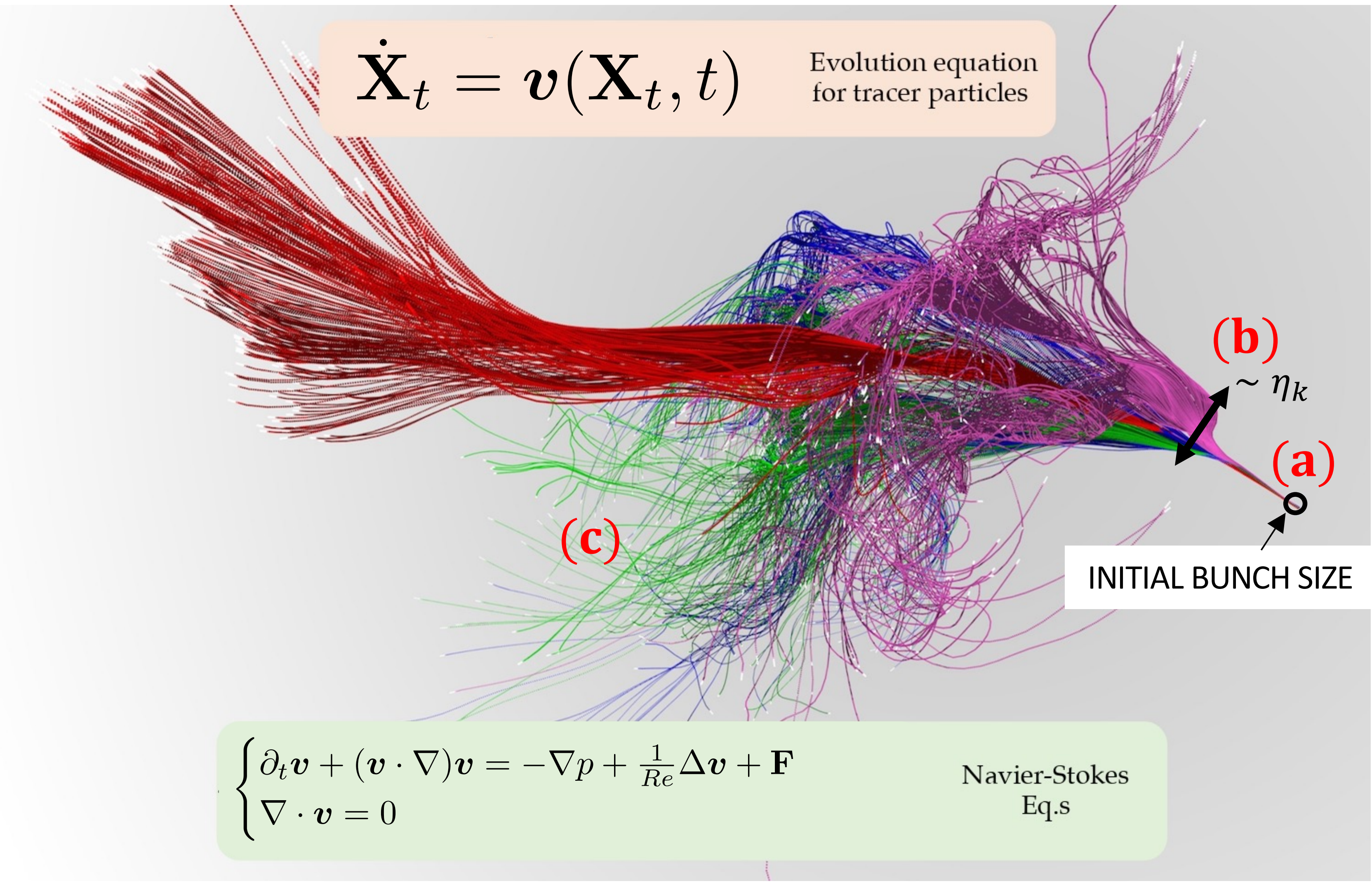
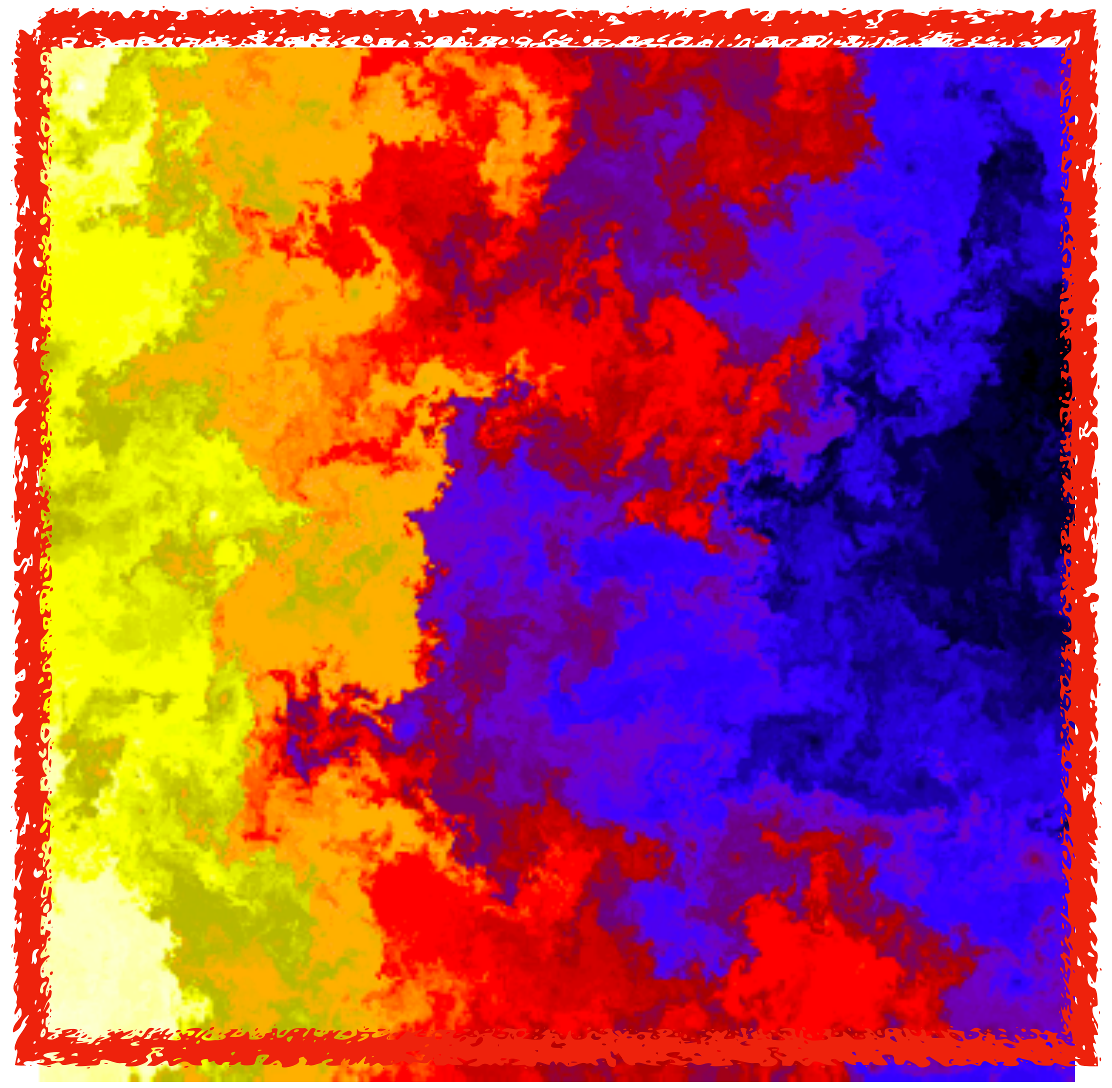






Outline -topic 3-

- Recall of the Eulerian view of scalar fields
- Lagrangian view of scalar transport: basic ideas
    - interpretation of dissipative anomaly
    - intermittency, origin of universality and zero modes
- Some extra (depending on time): active scalars

(Passive) Scalar Turbulence

Shraiman & Siggia, Nature **405**, 639 (2000)

velocity is given and not

modified by the transported field

$$\boxed{\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c}$$

Yaglom relation (similar to $4/5$-law)

$$\langle \delta_r v (\delta_r c)^2 \rangle = -4/3 \epsilon_c r$$

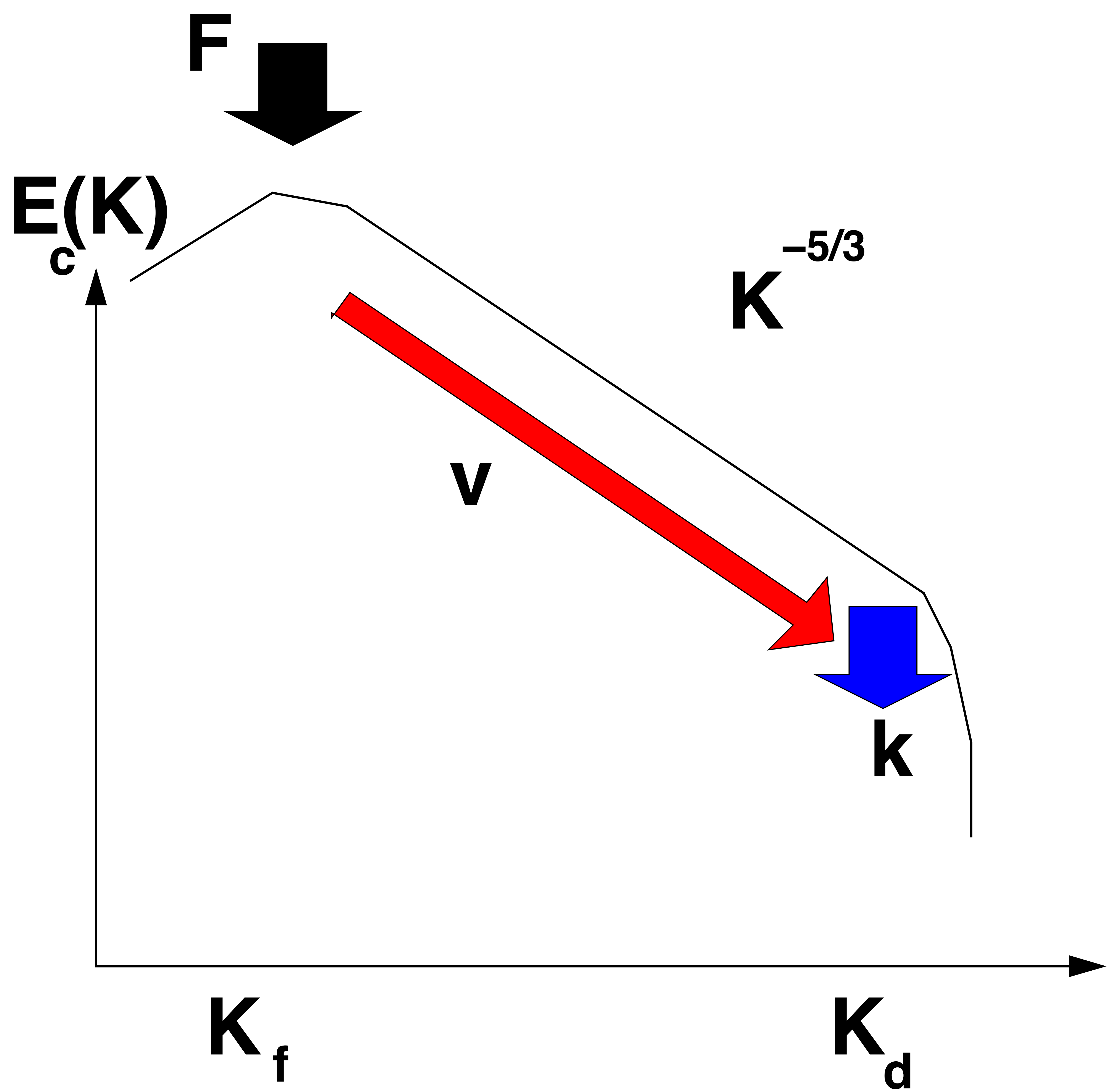
Phenomenology very similar to NS

- Cascade towards the small scales
  - Finite energy dissipation for $\kappa \to 0$
- Intermittency of the small scales

(Passive) Scalar Turbulence

$$\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta c + F_c$$

$$\frac{1}{2}\frac{d}{dt}\int d\boldsymbol{x}\, c^2(\boldsymbol{x},t) =< cF_c > -\kappa < |\boldsymbol{\nabla} c|^2 > = F_0 - \epsilon_c \approx 0$$



Dissipative anomaly $\langle \kappa |\boldsymbol{\nabla} c|^2 \rangle = \epsilon_c \approx \langle c F_c \rangle$

$$\lim_{\kappa \to 0} \epsilon_c \neq 0$$

Yaglom Relation

$$\langle \delta_r v (\delta_r c)^2 \rangle = -\frac{4}{3}\epsilon_c r$$

assume K41 turbulence $\delta_r v \sim (\epsilon_v r)^{1/3}$

by dimensional arguments $\delta_r c \sim \epsilon_c^{\frac{1}{2}} \epsilon_v^{-\frac{1}{6}} r^{\frac{1}{3}}$

(Passive) Scalar Turbulence

$$\langle (\delta_r c)^n \rangle = B_n (\epsilon_c^{1/2} \epsilon_v^{-1/6} r^{1/3})^n \left(\frac{L}{r}\right)^{n/3-\sigma_n} \qquad \sigma_n \neq \sigma_n^{dim} = \frac{n}{3}$$

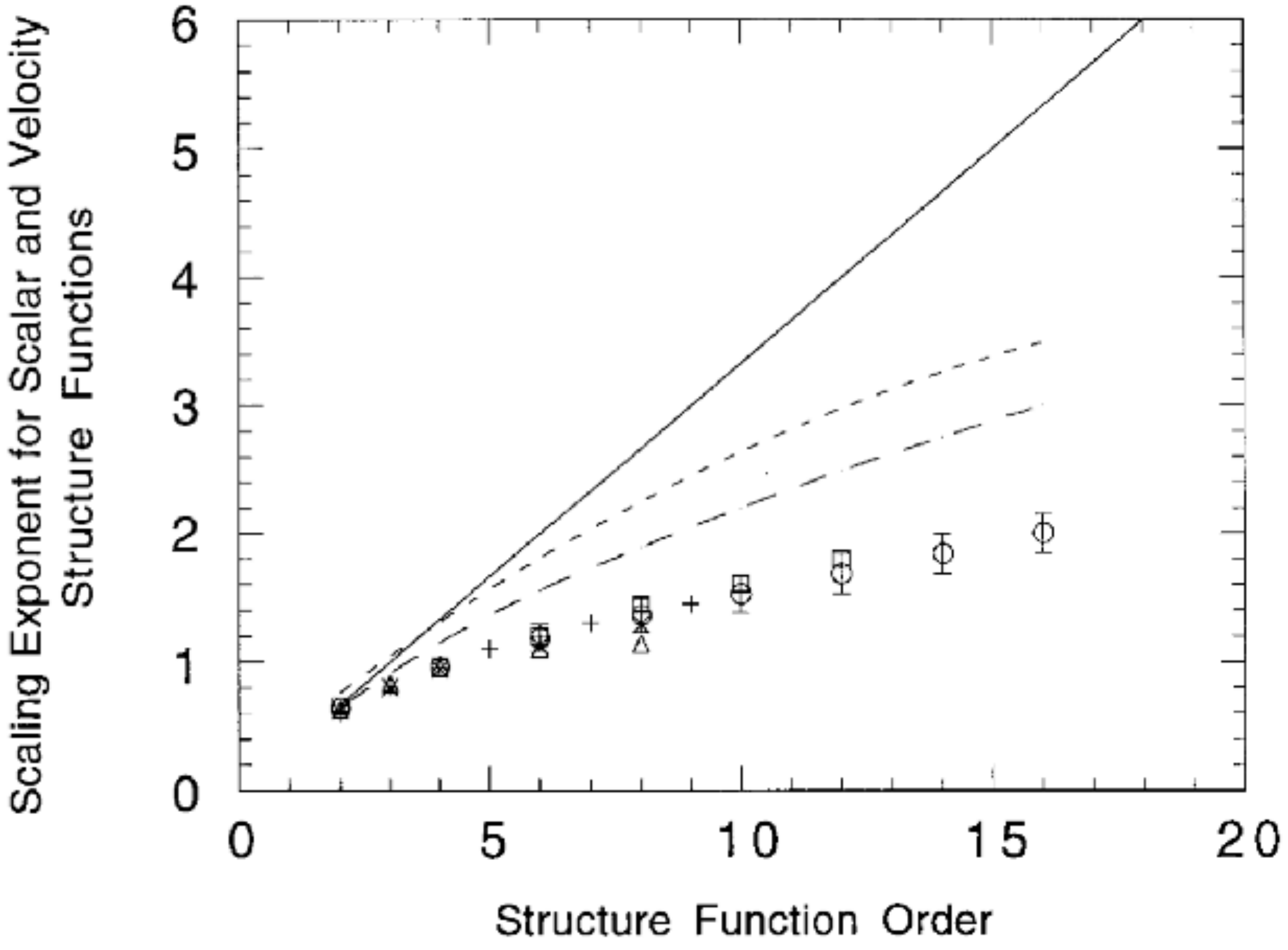


Figure 11 The scaling exponent ζ_n for the scalar structure function $\langle[\Delta\theta(r)]^n\rangle$ within the inertial subrange as a function of n. *Squares* are from the data of Antonia et al (1984) (heated jet), *crosses* are from the data of Ruiz-Chavarvia et al (1996) (heated wake), *triangles* are from the data of Meneveau et al (1990) (heated wake), *circles* are from the data of Mydlarski & Warhaft (1998a) (grid turbulence), and *plus signs* are from the full, three dimensional Navier-Stokes numerical simulations of Chen & Kraichnan (1998). *Vertical bars* represent uncertainty for the Mydlarski & Warhaft data. The *long-dashed line* is the white-noise estimate from Kraichnan (1994). *The short-dashed line* is for the velocity field from Anselmet (1984). The solid line is the KOC prediction.

Warhaft, Ann. Rev. Fluid Mech. **32**, 203 (2000)

IN 3D turbulent flows

Scalar increments display anomalous scaling and the exponents appear to be universal

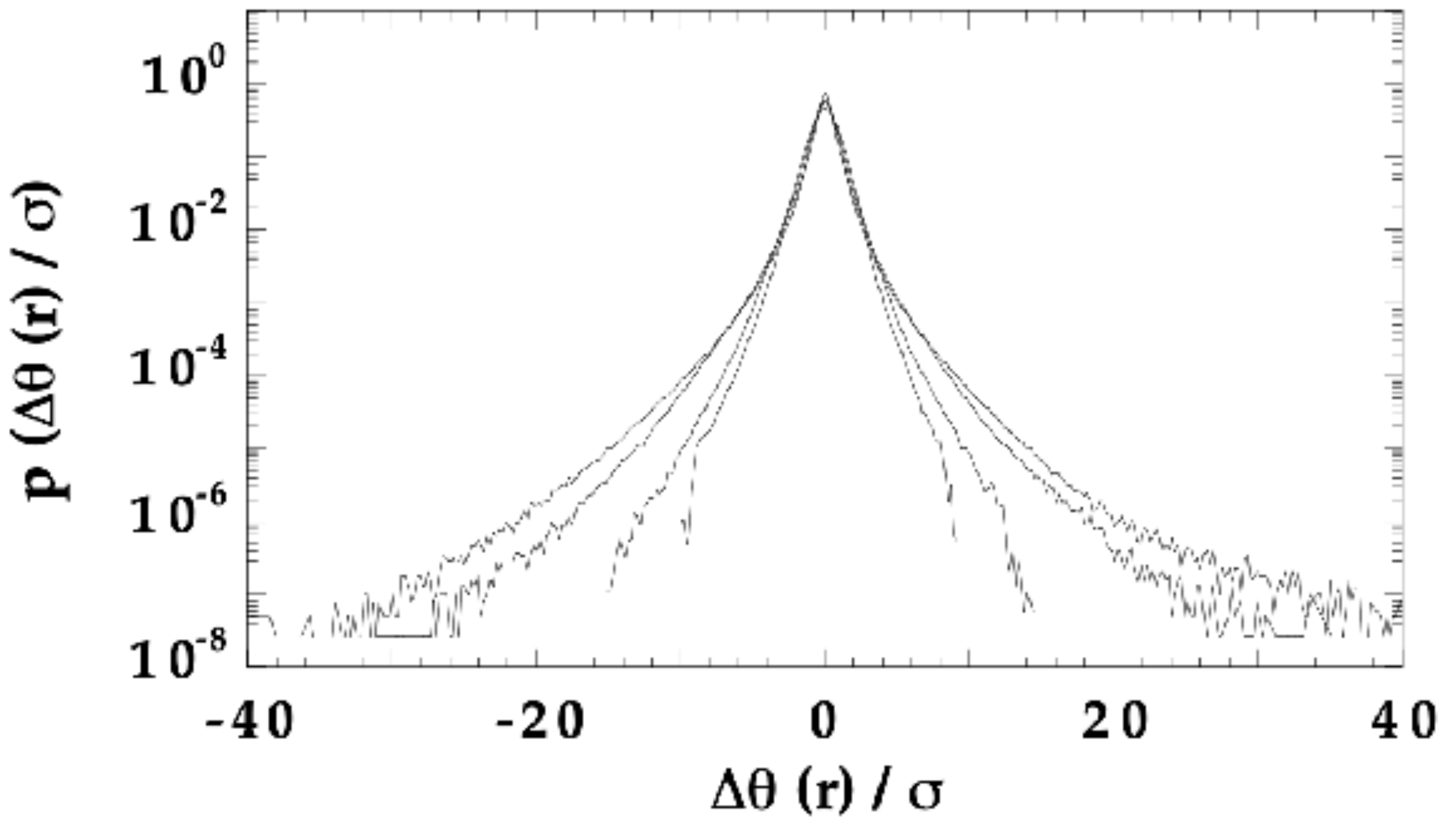


FIG. 2. Probability density functions of the normalized temperature increments, for $R_\lambda = 650$ (file #4 of Tab. I). From the inner to the outer pdf, $r/\eta = 10^4$, 600, 30 and 3.

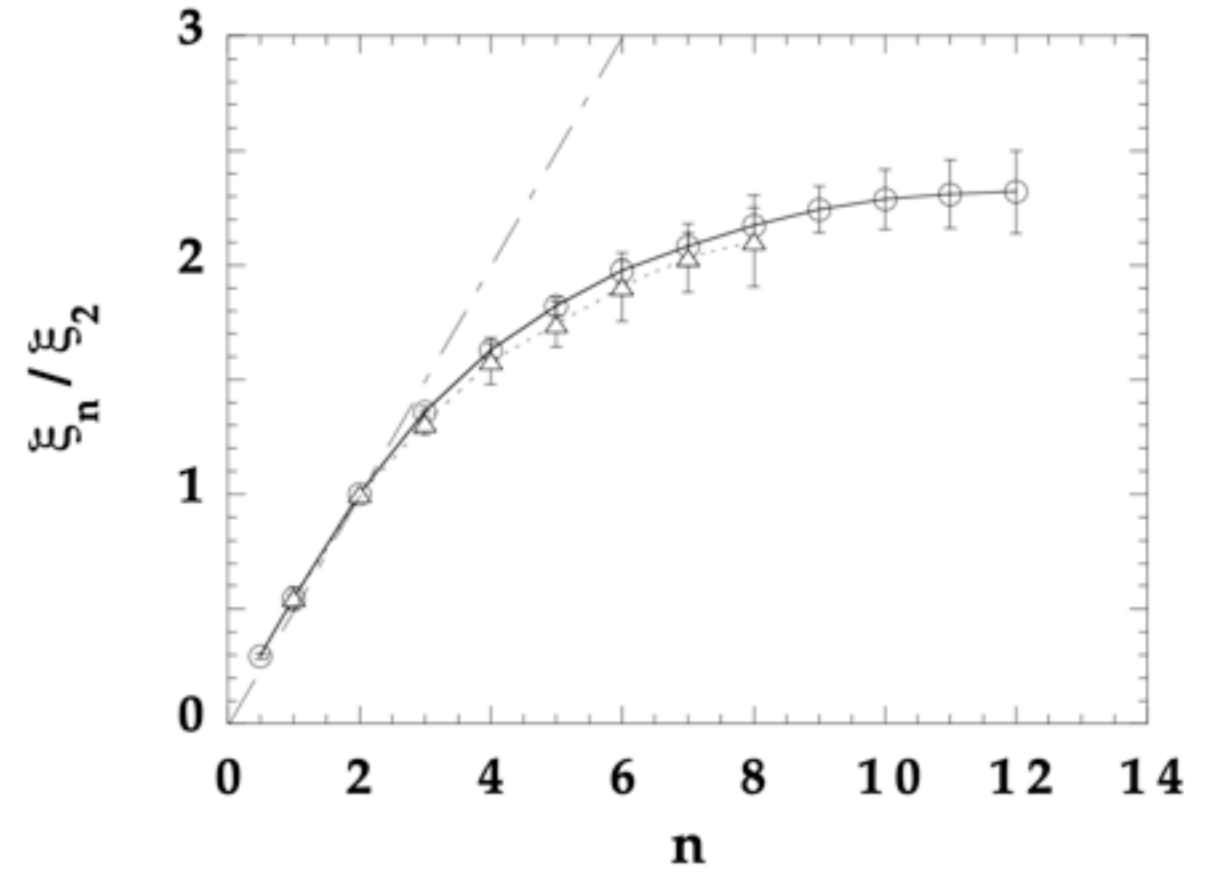


FIG. 3. Temperature structure function exponents, ξ_n, normalized by ξ_2. ○: $R_\lambda = 280$ in COR mode, △: $R_\lambda = 650$ in CTR mode. The dashed line indicates the Corrsin-Obukhov scaling $n/2$.

F. Moisy, H. Willaime, J.S. Andersen P. Tabeling
PRL 86, 4827 (2001)

(Passive) Scalar Turbulence

In 2D turbulence in the inverse cascade of velocity: the velocity field is non intermittent and display K41 scaling yet the scalar field is intermittent and the exponents are universal

So scalar intermittency is not inherited from velocity intermittency!!!

$$S_n(r) = \langle (\delta_r c)^n \rangle$$

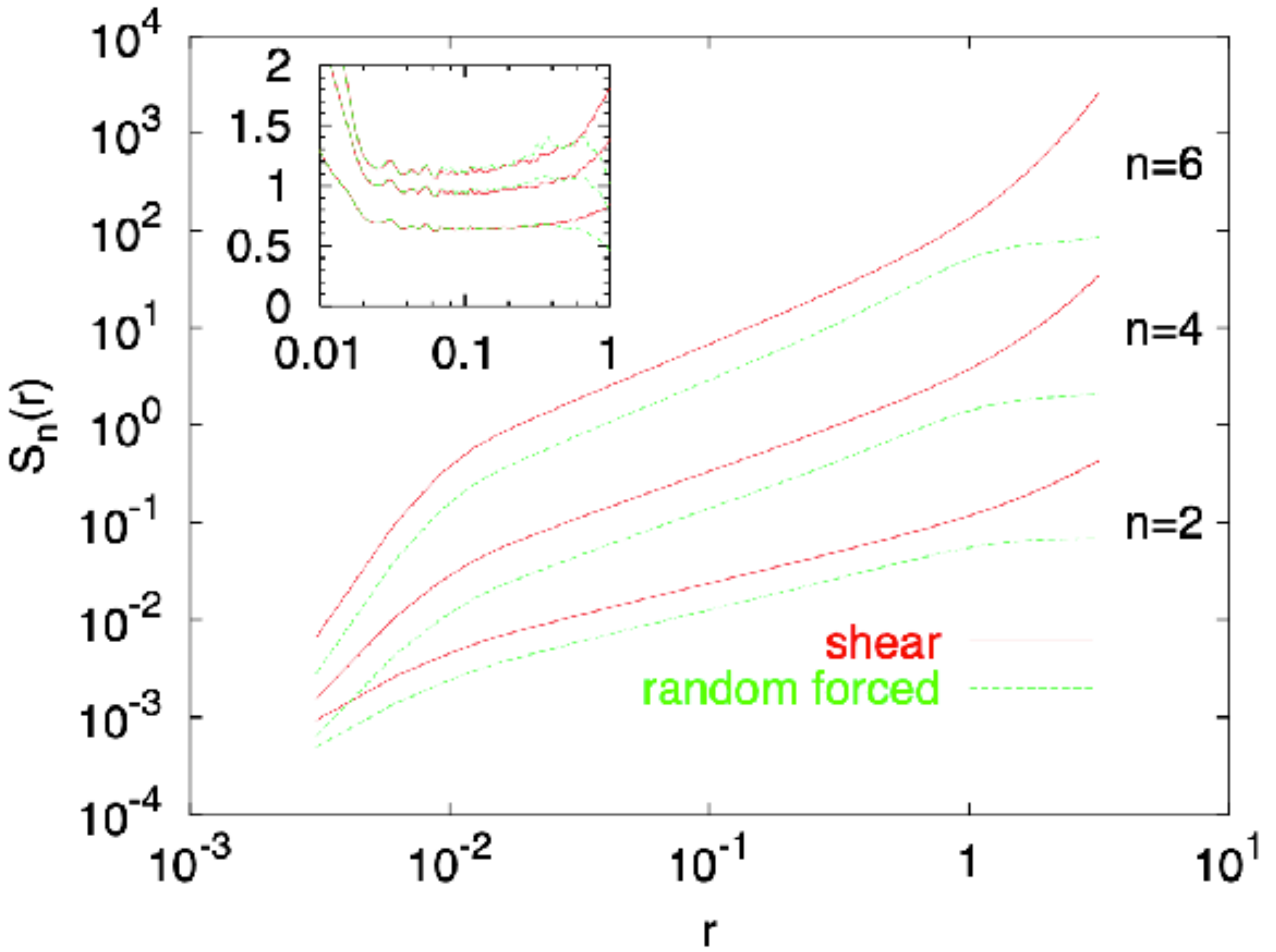


FIG. 4. Low-order even structure functions. Local scaling exponents are shown in the inset. The measured exponents are $\zeta_2 = 0.66 \pm 0.03$, $\zeta_4 = 0.95 \pm 0.04$ and $\zeta_6 = 1.11 \pm 0.04$.

$$P(\delta_r c)$$

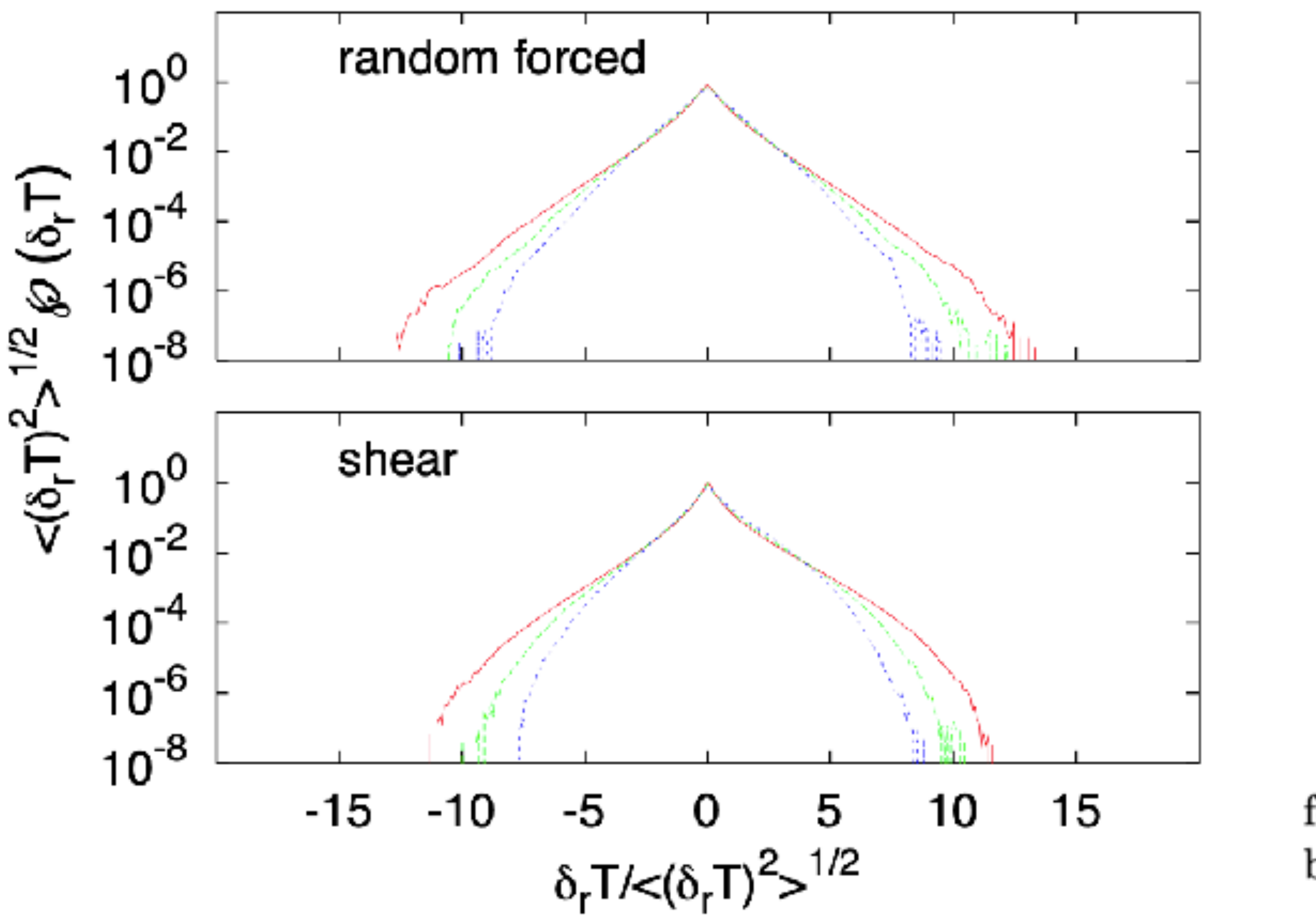


FIG. 3. Pdf's of scalar increments normalized by their standard deviations for three separations $r = 2.5\times 10^{-2}$, 5×10^{-2}, 10^{-1} in the inertial range.

$$S_n(r) = \langle (\delta_r c)^n \rangle$$

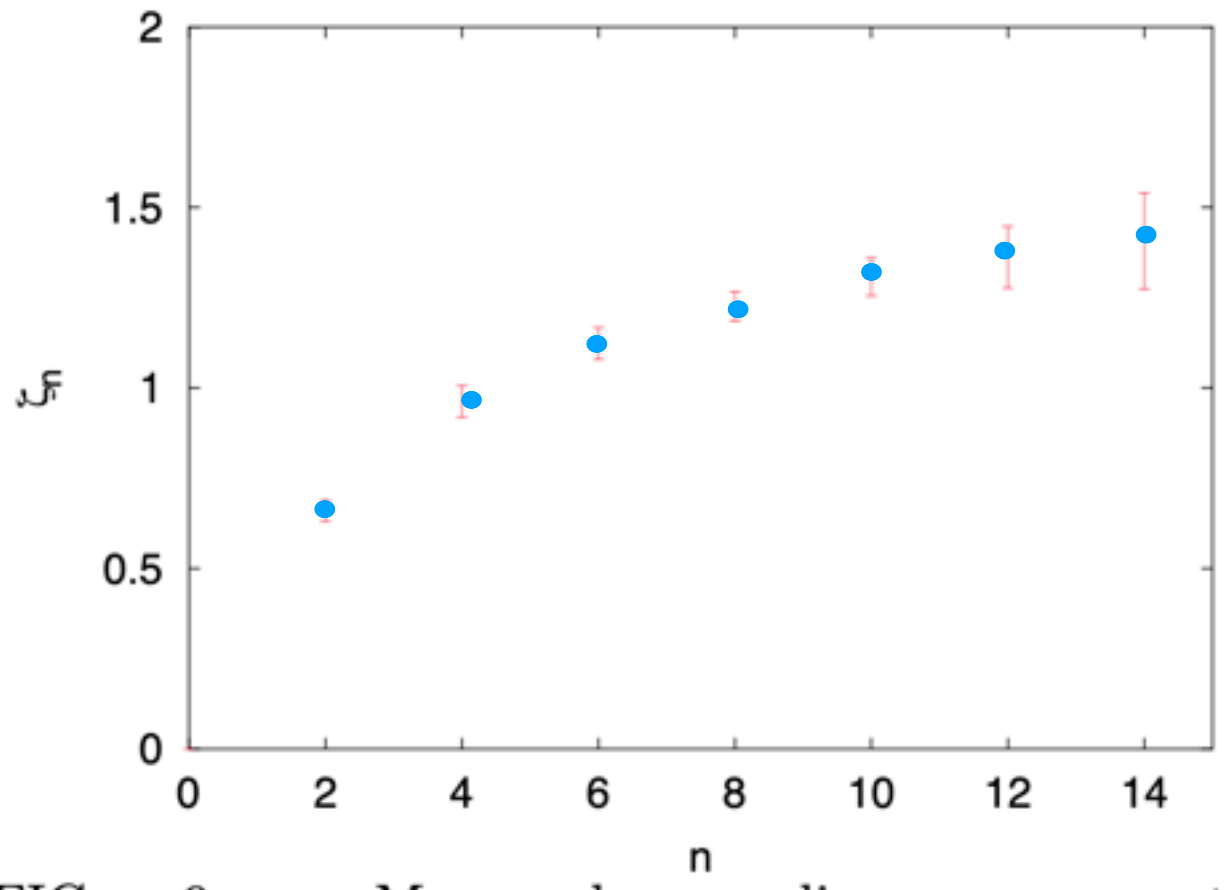


FIG. 9. Measured scaling exponent ζ_n for the Navier–Stokes advection. Error bars are estimated by the rms fluctuations of local scaling exponents.

A Celani, A Lanotte, A., Mazzino, M Vergassola, PRL *84*, 2385 (2000); & *PoF*, *13*, 1768-1783 (2001)

Lagrangian view of scalar transport

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = \kappa \Delta \theta + F$$

is equivalent to the SDE

$$\begin{aligned} \dot{\boldsymbol{y}}(s) &= \boldsymbol{v}(\boldsymbol{y}(s), s) + \sqrt{2\kappa}\eta(s) \qquad \langle \eta(t)\eta(t')\rangle = \delta(t-t') \\ \dot{\phi}(s) &= F(\boldsymbol{y}(s), s) \end{aligned}$$

$$\theta(\boldsymbol{x}, t) = \langle \phi \rangle_{\eta | \boldsymbol{y}(t) = \boldsymbol{x}} = \int^t ds \langle F(\boldsymbol{y}(s; \boldsymbol{x}, t), s) \rangle_\eta$$

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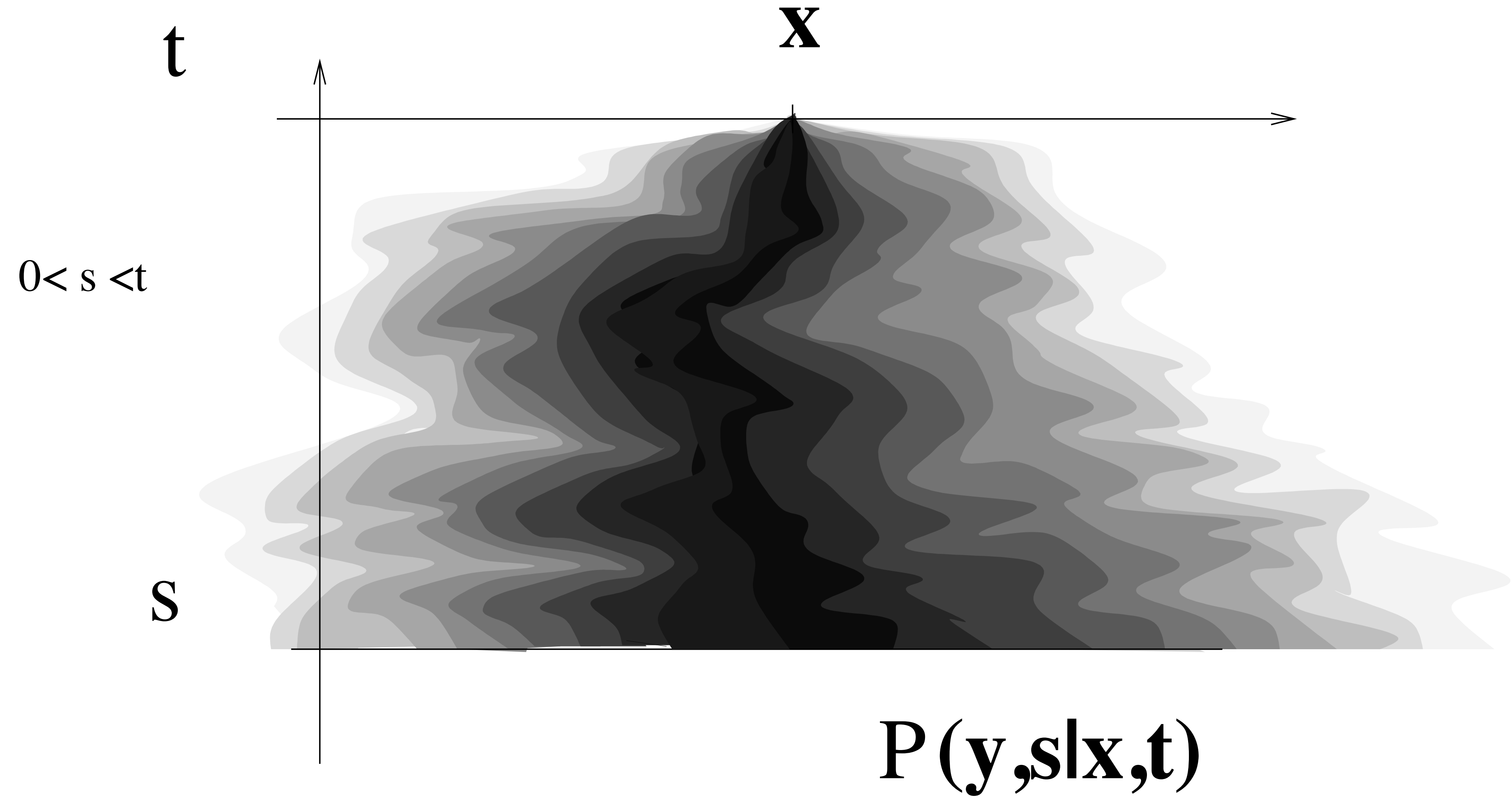
$$\theta(\boldsymbol{x}, t) = \langle \phi \rangle_{\eta | \boldsymbol{y}(t)=\boldsymbol{x}} = \int^t ds \langle F(\boldsymbol{y}(s; \boldsymbol{x}, t), s)\rangle_\eta$$

$$\theta(\boldsymbol{x}, t) = \int^t ds \int d\boldsymbol{y} p_v(\boldsymbol{y}, s|\boldsymbol{x}, t) F(\boldsymbol{y}, s)$$



Lagrangian propagator

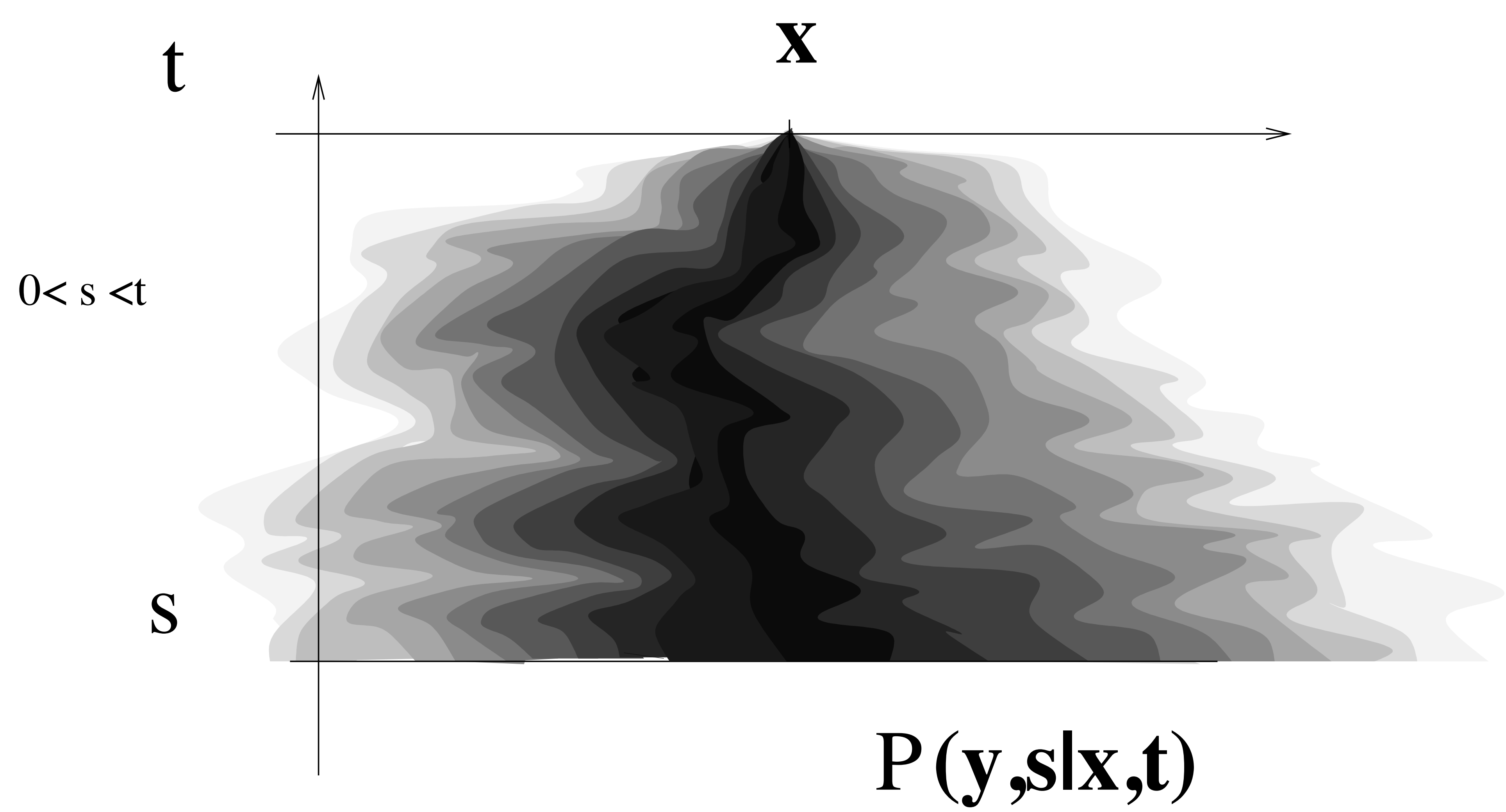
$$\begin{aligned} p_v(\boldsymbol{y}, t|\boldsymbol{x}, t) &= \delta(\boldsymbol{x} - \boldsymbol{y}) \\ p_v(\boldsymbol{y}, s|\boldsymbol{x}, t) &= \langle \delta(\boldsymbol{y} - \boldsymbol{y}(s; \boldsymbol{x}, t))\rangle_\eta \end{aligned}$$

Lagrangian view of scalar transport

$$\partial_t \theta + \boldsymbol{v}\cdot\boldsymbol{\nabla}\theta = \kappa\Delta\theta + F$$

is equivalent to the SDE

$$\begin{aligned}\dot{\boldsymbol{y}}(s) &= \boldsymbol{v}(\boldsymbol{y}(s),s) + \sqrt{2\kappa}\eta(s) \qquad \langle\eta(t)\eta(t')\rangle = \delta(t-t') \\ \dot{\phi}(s) &= F(\boldsymbol{y}(s),s)\end{aligned}$$



Lagrangian propagator

$$\theta(\boldsymbol{x},t) = \langle\phi\rangle_{\eta|\boldsymbol{y}(t)=\boldsymbol{x}} = \int^t ds\langle F(\boldsymbol{y}(s;\boldsymbol{x},t),s)\rangle_\eta$$

$$\begin{aligned}p_v(\boldsymbol{y},t|\boldsymbol{x},t) &= \delta(\boldsymbol{x}-\boldsymbol{y}) \\ p_v(\boldsymbol{y},s|\boldsymbol{x},t) &= \langle\delta(\boldsymbol{y}-\boldsymbol{y}(s;\boldsymbol{x},t))\rangle_\eta\end{aligned}$$

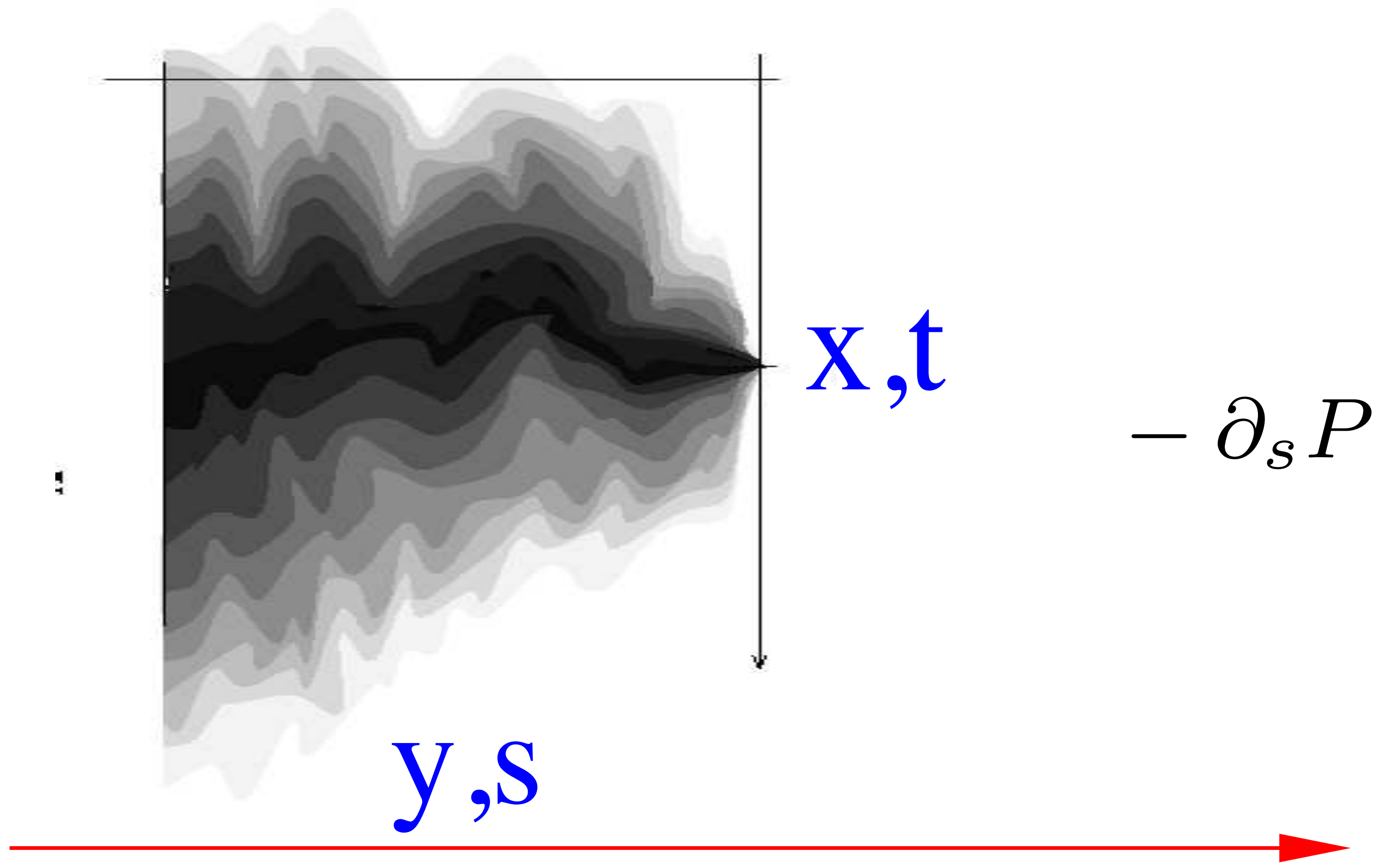
$$\theta(\boldsymbol{x},t) = \int^t ds \int d\boldsymbol{y} p_v(\boldsymbol{y},s|\boldsymbol{x},t)F(\boldsymbol{y},s) \quad \Rightarrow \quad \partial_t \theta + \boldsymbol{v}\cdot\boldsymbol{\nabla}\theta = \kappa\Delta\theta + F$$

$$\partial_t P(\boldsymbol{y},s|\boldsymbol{x},t) + \boldsymbol{\nabla}_x\cdot[\boldsymbol{v}(\boldsymbol{x},t)P(\boldsymbol{y},s|\boldsymbol{x},t)] = \kappa\Delta_x P(\boldsymbol{y},s|\boldsymbol{x},t)$$

Note that in passive scalars the propagator and the forcing are independent

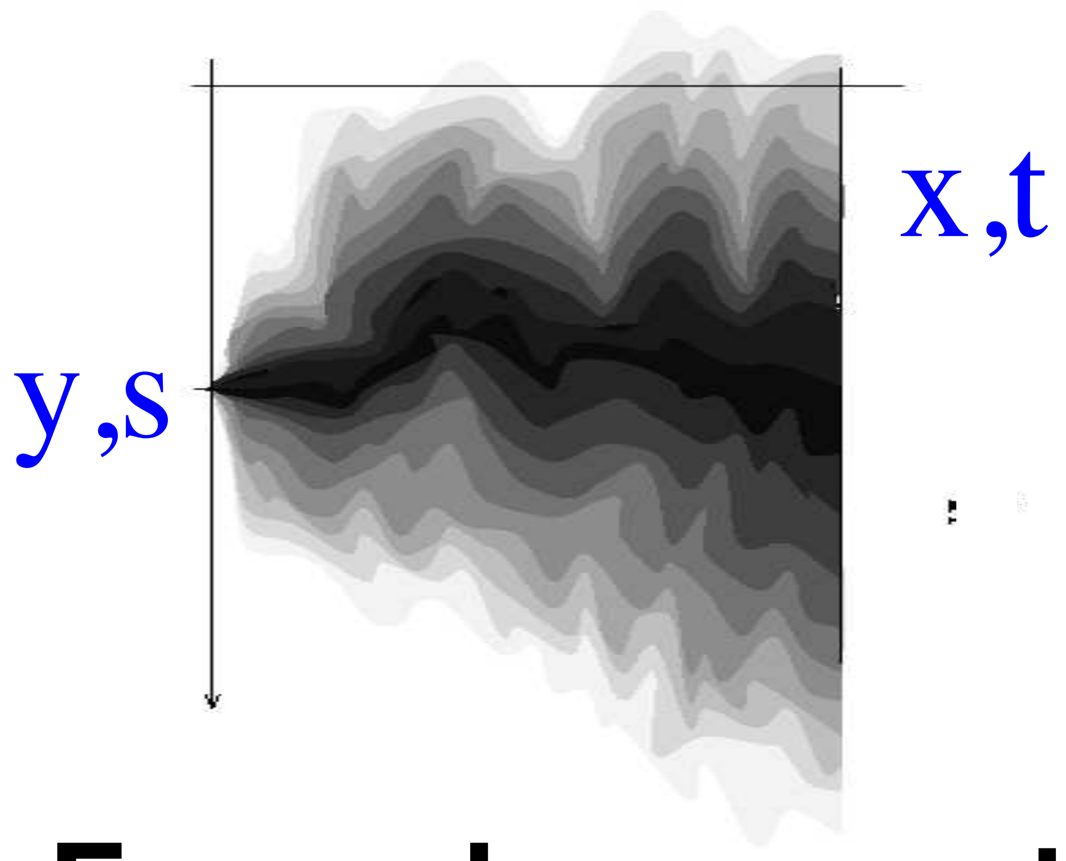
Backward & Forward propagator

Backward propagator



$$-\partial_s P(\boldsymbol{y}, s|\boldsymbol{x}, t) - \boldsymbol{v}(\boldsymbol{y}, s) \cdot \boldsymbol{\nabla}_y P(\boldsymbol{y}, s|\boldsymbol{x}, t) = \kappa \Delta_y P(\boldsymbol{y}, s|\boldsymbol{x}, t)$$

$$P(\boldsymbol{y}, s = t|\boldsymbol{x}, t) = \delta(\boldsymbol{x} - \boldsymbol{y})$$

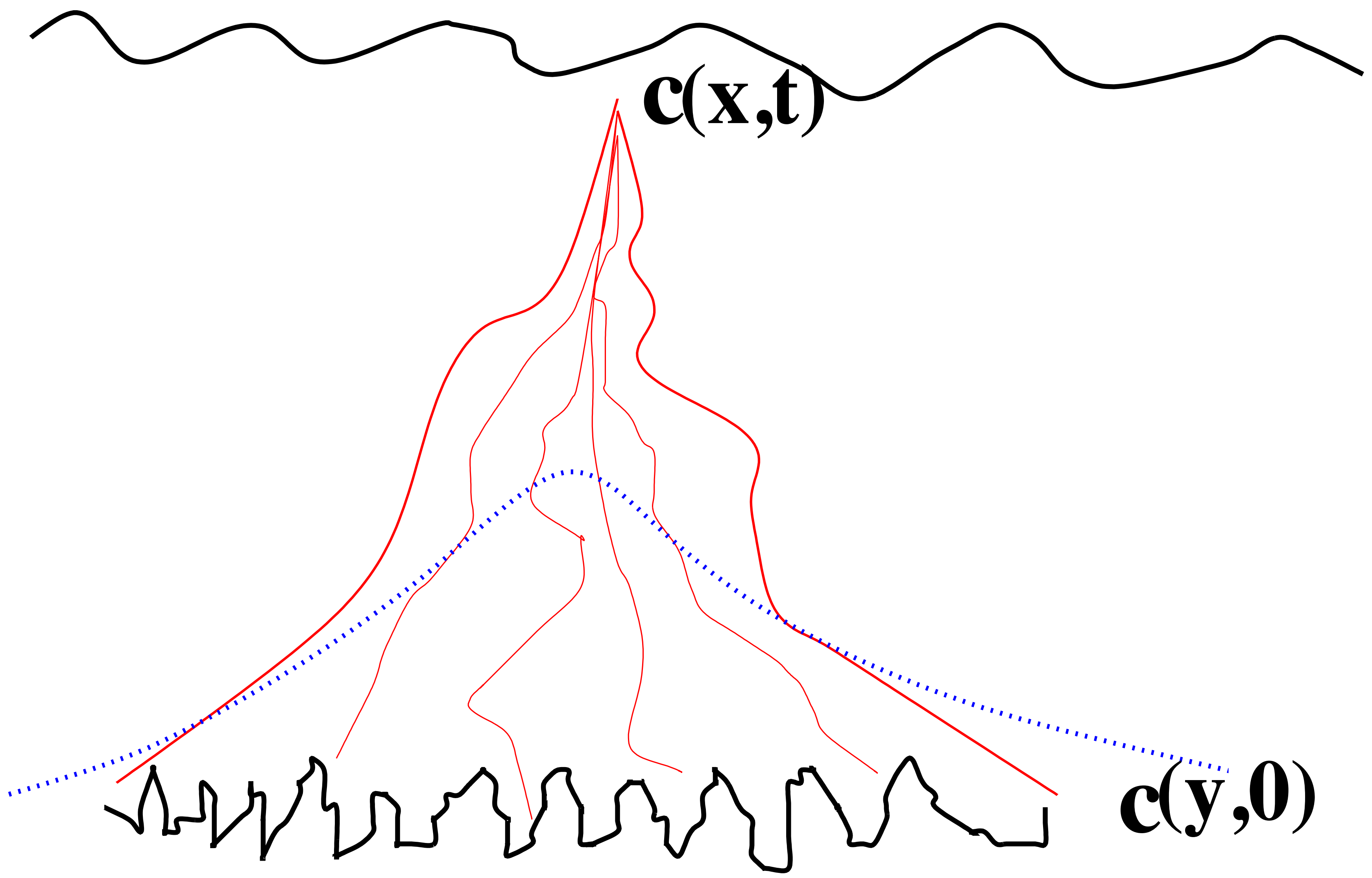


$$\partial_t P(\boldsymbol{y}, s|\boldsymbol{x}, t) + \boldsymbol{\nabla}_x \cdot [\boldsymbol{v}(\boldsymbol{x}, t) P(\boldsymbol{y}, s|\boldsymbol{x}, t)] = \kappa \Delta_x P(\boldsymbol{y}, s|\boldsymbol{x}, t)$$

Forwad propagator

Explosive separation $\Longrightarrow$ dissipative anomaly

Non-uniqueness of paths $\Longrightarrow$ dissipative anomaly



$$\frac{d}{dt}c^2(t) = -\epsilon_c$$

$$\kappa \longrightarrow 0 \qquad \epsilon_c \neq 0$$

non-smooth flows Since the scalar forcing and the Lagrangian paths are uncorrelated the existence of many paths implies the blurring of the initial scalar field as times goes on

smooth flows Lagrangian paths are unique and for dissipation to take place a non zero molecular diffusivity is needed

Explosive separation $\Longrightarrow$ dissipative anomaly

Eulerian

$$\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta\, c + f_c$$

$$\langle f_c(\boldsymbol{x}_1, t) f_c(\boldsymbol{x}_2, t') \rangle = \delta(t - t') \mathcal{F}(|\boldsymbol{x}_1 - \boldsymbol{x}_2| / \ell_f)$$

Lagrangian

$$\frac{\mathrm{d}\boldsymbol{\rho}(s)}{\mathrm{d}s} = \boldsymbol{v}(\boldsymbol{\rho}(s), s) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}}(s), \qquad \boldsymbol{\rho}(t) = \boldsymbol{x} \qquad \frac{\mathrm{d}\phi^{\boldsymbol{w}}(s)}{\mathrm{d}s} = f_c(\boldsymbol{\rho}(s), s)$$

$$c(\boldsymbol{x}, t) = \langle \phi^{\boldsymbol{w}}(t) \rangle_{\boldsymbol{w}} = \left\langle \int_0^t \mathrm{d}s\, f_c(\boldsymbol{\rho}(s), s) \right\rangle_{\boldsymbol{w}}.$$

$$\langle c^2(\boldsymbol{x}, t) \rangle = \left\langle \int_0^t \int_0^t \mathrm{d}s_1\, \mathrm{d}s_2\, f_c(\boldsymbol{\rho}(s_1; \boldsymbol{x}, t))\, f_c(\boldsymbol{\rho}(s_2; \boldsymbol{x}, t)) \right\rangle = \left\langle \left(\int_0^t \mathrm{d}s f_c(\boldsymbol{\rho}(s; \boldsymbol{x}, t)) \right)^2 \right\rangle$$

due to delta correlation we may think $\langle c^2(x, t) \rangle \propto t$

Explosive separation $\Longrightarrow$ dissipative anomaly

Eulerian

$$\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta\, c + f_c$$

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Lagrangian

$$\frac{\mathrm{d}\boldsymbol{\rho}(s)}{\mathrm{d}s} = \boldsymbol{v}(\boldsymbol{\rho}(s), s) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}}(s), \qquad \boldsymbol{\rho}(t) = \boldsymbol{x} \qquad \frac{\mathrm{d}\phi^{\boldsymbol{w}}(s)}{\mathrm{d}s} = f_c(\boldsymbol{\rho}(s), s)$$

$$c(\boldsymbol{x}, t) = \langle \phi^{\boldsymbol{w}}(t)\rangle_{\boldsymbol{w}} = \left\langle \int_0^t \mathrm{d}s\, f_c(\boldsymbol{\rho}(s), s) \right\rangle_{\boldsymbol{w}}.$$

$$\langle c^2(\boldsymbol{x}, t)\rangle = \left\langle \int_0^t \int_0^t \mathrm{d}s_1\, \mathrm{d}s_2\, f_c(\boldsymbol{\rho}(s_1; \boldsymbol{x}, t))\, f_c(\boldsymbol{\rho}(s_2; \boldsymbol{x}, t)) \right\rangle = \left\langle \left(\int_0^t \mathrm{d}s f_c(\boldsymbol{\rho}(s; \boldsymbol{x}, t)) \right)^2 \right\rangle$$

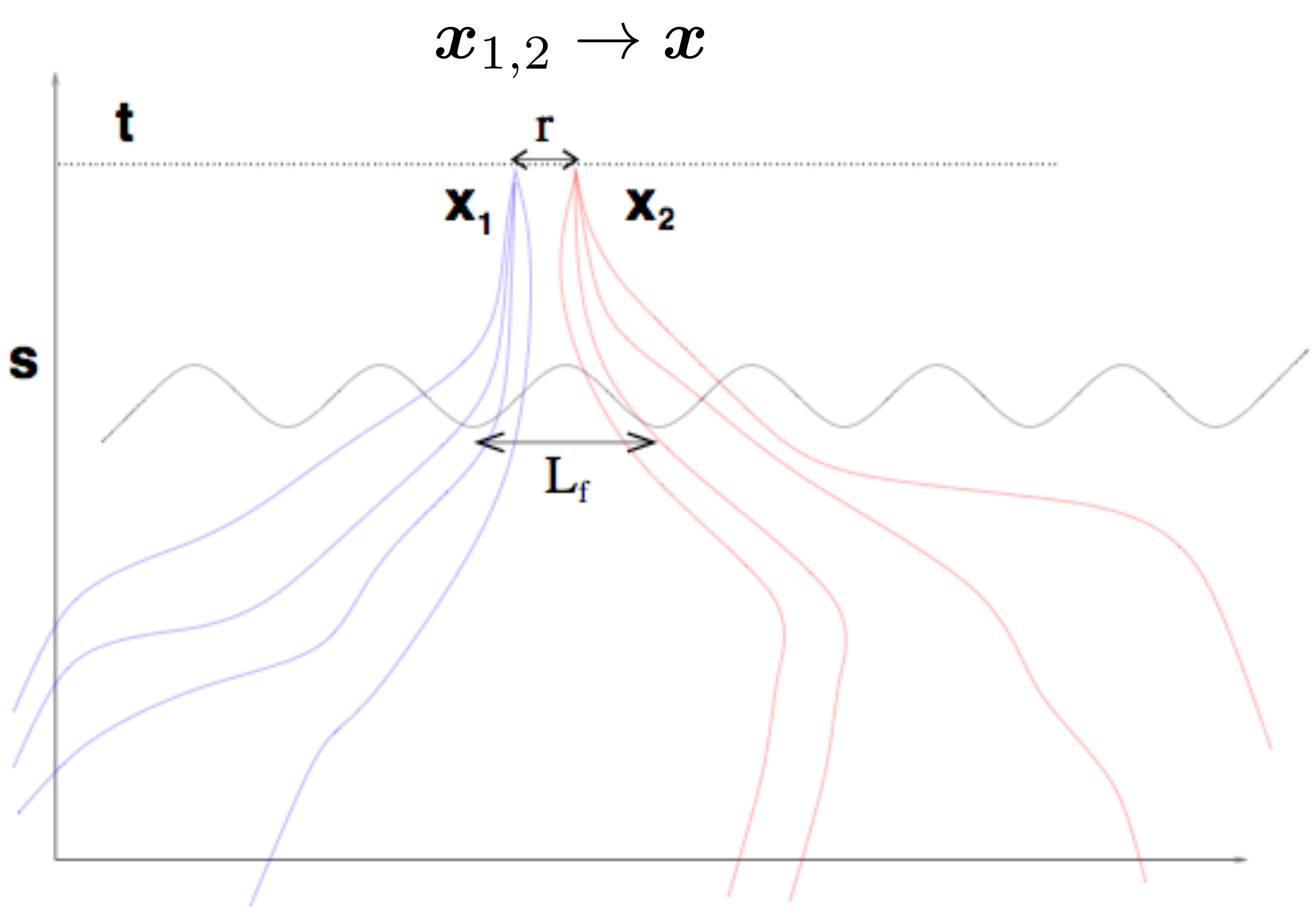
due to delta correlation we may think $\langle c^2(x,t)\rangle \propto t$

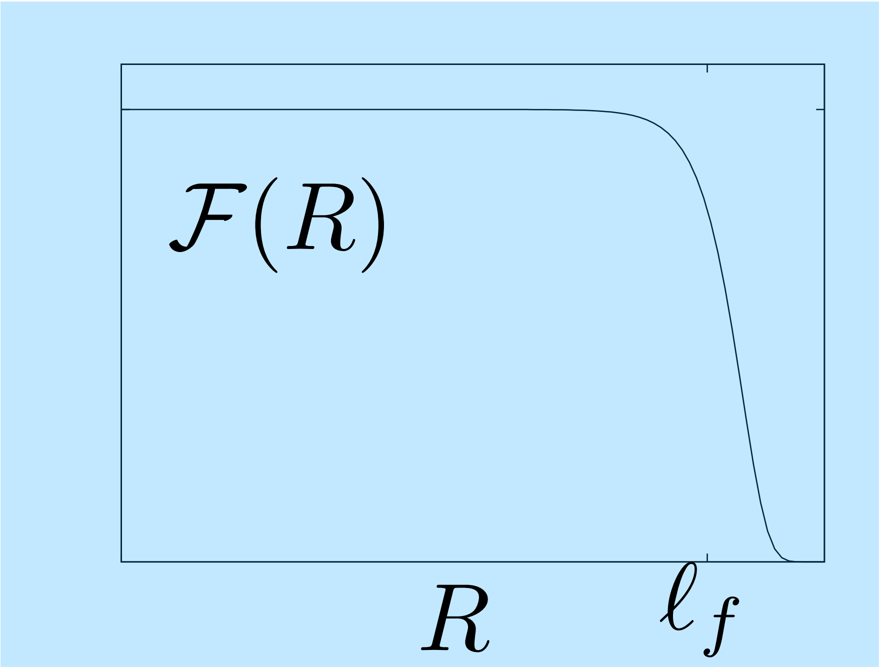
But let us rewrite it as

$$\langle c^2(\boldsymbol{x}, t)\rangle = \int_0^t \mathrm{d}s \iint \langle P_2(\boldsymbol{y}_1, \boldsymbol{y}_2, s | \boldsymbol{x}, \boldsymbol{x}, t)\rangle_v \mathcal{F}(|\boldsymbol{y}_1 - \boldsymbol{y}_2|/\ell_f)\, \mathrm{d}\boldsymbol{y}_1\, \mathrm{d}\boldsymbol{y}_2.$$

Explosive separation $\Longrightarrow$ dissipative anomaly

$$\langle c^2(\boldsymbol{x},t)\rangle = \int_0^t \mathrm{d}s \iint \langle P_2(\boldsymbol{y}_1,\boldsymbol{y}_2,s|\boldsymbol{x},\boldsymbol{x},t)\rangle_v \mathcal{F}(|\boldsymbol{y}_1-\boldsymbol{y}_2|/\ell_f)\,\mathrm{d}\boldsymbol{y}_1\,\mathrm{d}\boldsymbol{y}_2.$$





The time integral is cut off at $|t-s| \gg \mathcal{T}_{\ell_f}$

which is the time for two coinciding paths to separate (backward in time) to a distance of order ℓ_f

non uniqueness $\rightarrow$ dissipative anomaly

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v} \cdot \nabla \Theta = \kappa \Delta \Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t), t) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}},$$

$$\frac{\mathrm{d}\vartheta(t)}{\mathrm{d}t} = f_\Theta(\boldsymbol{\rho}(t), t).$$

dissipative anomaly (refined argument)

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$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t), t) + \sqrt{2\kappa}\dot{\boldsymbol{w}},$$

$$\frac{\mathrm{d}\vartheta(t)}{\mathrm{d}t} = f_\Theta(\boldsymbol{\rho}(t), t).$$

But now we do not condition on the final position, let us consider all paths and averaging over those landing in x,t we have

$$\Theta(\boldsymbol{x}, t) = \langle \vartheta(t) \rangle_{\boldsymbol{w}}$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v} \cdot \nabla \Theta = \kappa \Delta \Theta + f_\Theta$$

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But now we do not condition on the final position, let us consider all paths and averaging over those landing in x,t we have

$$\Theta(\boldsymbol{x}, t) = \langle \vartheta(t) \rangle_{\boldsymbol{w}}$$

Be $P(\boldsymbol{x}, \vartheta, t | \boldsymbol{x}_0, \vartheta_0, 0)$ **the prob to start in** x_0 **with** $\vartheta_0 = \Theta(x_0, 0)$ **and to land in x at time t carrying a scalar value** ϑ

$$\partial_t P + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} P + f_\Theta \nabla_\vartheta P = \kappa \Delta P$$

$$P(\boldsymbol{x}, \vartheta, t = 0 | \boldsymbol{x}_0, \vartheta_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(\vartheta - \vartheta_0)$$

$$\vartheta_0 = \Theta(\boldsymbol{x}, 0)$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v} \cdot \nabla \Theta = \kappa \Delta \Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t), t) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}},$$

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$$\partial_t P + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} P + f_\Theta \nabla_\vartheta P = \kappa \Delta P$$

$$P(\boldsymbol{x}, \vartheta, t|\boldsymbol{x}_0, \vartheta_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(\vartheta - \vartheta_0)$$

$$\vartheta_0 = \Theta(\boldsymbol{x}, 0)$$

We average over the initial conditions

$$\mathcal{P}(\boldsymbol{x}, \vartheta, t) = \int P(\boldsymbol{x}, \vartheta, t|\boldsymbol{x}_0, \Theta(\boldsymbol{x}_0, 0), 0)\, \mathrm{d}\boldsymbol{x}_0$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v} \cdot \nabla \Theta = \kappa \Delta \Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t), t) + \sqrt{2\kappa}\dot{\boldsymbol{w}},$$

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$$\partial_t P + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} P + f_\Theta \nabla_\vartheta P = \kappa \Delta P$$

$$P(\boldsymbol{x}, \vartheta, t|\boldsymbol{x}_0, \vartheta_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(\vartheta - \vartheta_0) \quad t=0$$

$$\vartheta_0 = \Theta(\boldsymbol{x}, 0)$$

We average over the initial conditions

$$\mathcal{P}(\boldsymbol{x}, \vartheta, t) = \int P(\boldsymbol{x}, \vartheta, t|\boldsymbol{x}_0, \Theta(\boldsymbol{x}_0, 0), 0)\, \mathrm{d}\boldsymbol{x}_0$$

$$\vartheta(t) = \int_0^t f_\Theta(\rho(s), s)\, \mathrm{d}s$$

$$\sigma_\Theta^2(\boldsymbol{x}, t) = \int \vartheta^2 \mathcal{P}\, \mathrm{d}\vartheta - \left(\int \vartheta \mathcal{P}\, \mathrm{d}\vartheta\right)^2$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v} \cdot \nabla \Theta = \kappa \Delta \Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t), t) + \sqrt{2\kappa}\dot{\boldsymbol{w}},$$

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But now we do not condition on the final position, let us consider all paths and averaging over those landing in x,t we have

$$\Theta(\boldsymbol{x}, t) = \langle \vartheta(t) \rangle_{\boldsymbol{w}}$$

Be $P(\boldsymbol{x}, \vartheta, t|\boldsymbol{x}_0, \vartheta_0, 0)$ **the prob to start in** x_0 **with** $\vartheta_0 = \Theta(x_0,0)$ **and to land in x at time t carrying a scalar value** ϑ

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We average over the initial conditions

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$$\vartheta(t) = \int_0^t f_\Theta(\rho(s), s)\, \mathrm{d}s$$

$$\sigma^2_\Theta(\boldsymbol{x}, t) = \int \vartheta^2 \mathcal{P}\, \mathrm{d}\vartheta - \left(\int \vartheta \mathcal{P}\, \mathrm{d}\vartheta\right)^2$$

$$\partial_t \sigma^2_\Theta(\boldsymbol{x}, t) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \sigma^2_\Theta(\boldsymbol{x}, t) = \kappa \Delta \sigma^2_\Theta(\boldsymbol{x}, t) + 2\epsilon_\Theta(\boldsymbol{x}, t)$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v}\cdot\nabla\Theta = \kappa\Delta\Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t),t) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}},$$

$$\frac{\mathrm{d}\vartheta(t)}{\mathrm{d}t} = f_\Theta(\boldsymbol{\rho}(t),t).$$

But now we do not condition on the final position, let us consider all paths and averaging over those landing in x,t we have

$$\Theta(\boldsymbol{x},t) = \langle\vartheta(t)\rangle_{\boldsymbol{w}}$$

Be $P(\boldsymbol{x},\vartheta,t|\boldsymbol{x}_0,\vartheta_0,0)$ **the prob to start in** x_0 **with** $\vartheta_0 = \Theta(x_0{,}0)$ **and to land in x at time t carrying a scalar value** ϑ

$$\partial_t P + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}P + f_\Theta\nabla_\vartheta P = \kappa\Delta P$$

$$P(\boldsymbol{x},\vartheta,t|\boldsymbol{x}_0,\vartheta_0,0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(\vartheta-\vartheta_0) \quad (t=0)$$

$$\vartheta_0 = \Theta(\boldsymbol{x},0)$$

We average over the initial conditions

$$\mathcal{P}(\boldsymbol{x},\vartheta,t) = \int P(\boldsymbol{x},\vartheta,t|\boldsymbol{x}_0,\Theta(\boldsymbol{x}_0,0),0)\,\mathrm{d}\boldsymbol{x}_0$$

$$\vartheta(t) = \int_0^t f_\Theta(\rho(s),s)\,\mathrm{d}s$$

$$\sigma^2_\Theta(\boldsymbol{x},t) = \int\vartheta^2\mathcal{P}\,\mathrm{d}\vartheta - \left(\int\vartheta\mathcal{P}\,\mathrm{d}\vartheta\right)^2$$

$$\partial_t\sigma^2_\Theta(\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\sigma^2_\Theta(\boldsymbol{x},t) = \kappa\Delta\sigma^2_\Theta(\boldsymbol{x},t) + 2\epsilon_\Theta(\boldsymbol{x},t)$$

$$\epsilon_\Theta(\boldsymbol{x},t) = \kappa\left|\nabla_{\boldsymbol{x}}\int\vartheta\mathcal{P}(\boldsymbol{x},\vartheta,t)\,\mathrm{d}\vartheta\right|^2$$

dissipative anomaly (refined argument)

$$\partial_t \Theta + \boldsymbol{v}\cdot\nabla\Theta = \kappa\Delta\Theta + f_\Theta$$

$$\frac{\mathrm{d}\boldsymbol{\rho}(t)}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{\rho}(t),t) + \sqrt{2\kappa}\,\dot{\boldsymbol{w}},$$

$$\frac{\mathrm{d}\vartheta(t)}{\mathrm{d}t} = f_\Theta(\boldsymbol{\rho}(t),t).$$

But now we do not condition on the final position, let us consider all paths and averaging over those landing in x,t we have

$$\Theta(\boldsymbol{x},t) = \langle\vartheta(t)\rangle_{\boldsymbol{w}}$$

Be $P(\boldsymbol{x},\vartheta,t|\boldsymbol{x}_0,\vartheta_0,0)$ **the prob to start in** x_0 **with** $\vartheta_0 = \Theta(x_0,0)$ **and to land in x at time t carrying a scalar value** ϑ

$$\partial_t P + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}P + f_\Theta\nabla_\vartheta P = \kappa\Delta P$$

$$P(\boldsymbol{x},\vartheta,t_0|\boldsymbol{x}_0,\vartheta_0,0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(\vartheta-\vartheta_0)$$

$$\vartheta_0 = \Theta(\boldsymbol{x},0)$$

We average over the initial conditions

$$\mathcal{P}(\boldsymbol{x},\vartheta,t) = \int P(\boldsymbol{x},\vartheta,t|\boldsymbol{x}_0,\Theta(\boldsymbol{x}_0,0),0)\,\mathrm{d}\boldsymbol{x}_0$$

$$\vartheta(t) = \int_0^t f_\Theta(\rho(s),s)\,\mathrm{d}s$$

$$\sigma_\Theta^2(\boldsymbol{x},t) = \int \vartheta^2\mathcal{P}\,\mathrm{d}\vartheta - \left(\int\vartheta\mathcal{P}\,\mathrm{d}\vartheta\right)^2$$

$$\partial_t\sigma_\Theta^2(\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\sigma_\Theta^2(\boldsymbol{x},t) = \kappa\Delta\sigma_\Theta^2(\boldsymbol{x},t) + 2\epsilon_\Theta(\boldsymbol{x},t)$$

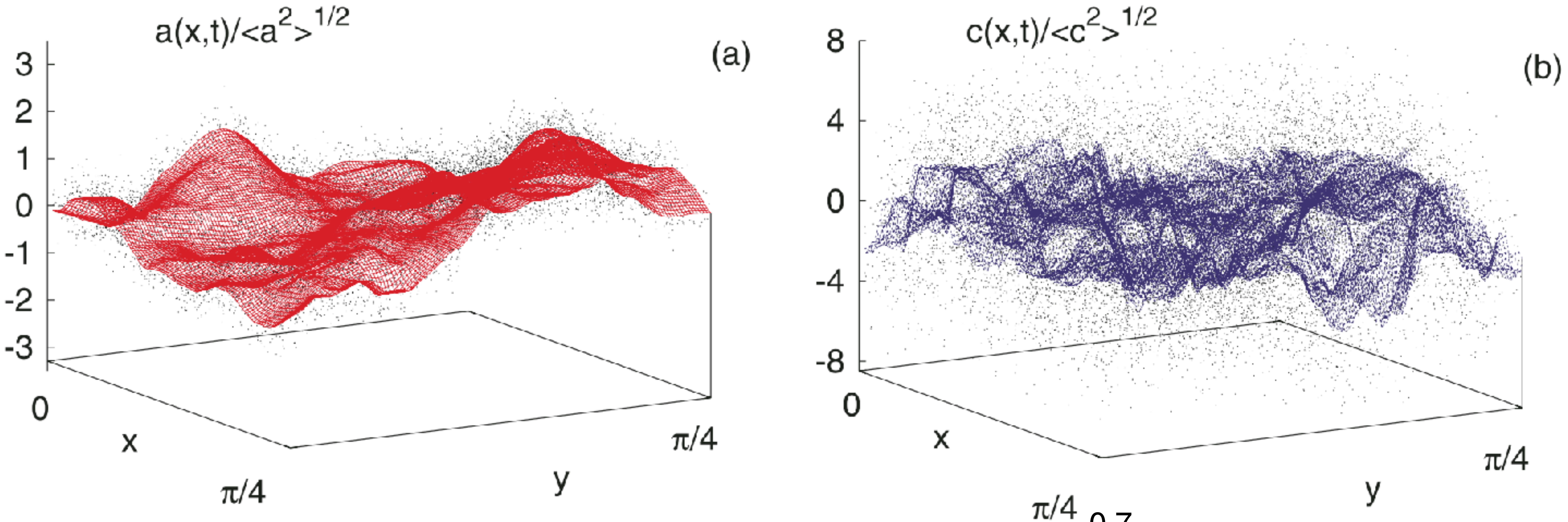
$$\epsilon_\Theta(\boldsymbol{x},t) = \kappa\left|\nabla_{\boldsymbol{x}}\int\vartheta\mathcal{P}(\boldsymbol{x},\vartheta,t)\,\mathrm{d}\vartheta\right|^2$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int\sigma_\Theta^2(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = 2\int\epsilon_\Theta(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = 2\epsilon_\Theta}$$

A Celani, M. C., A Mazzino, and M Vergassola. *New Journal of Physics* 6, (2004): 72.

see also T. D. Drivas & G L. Eyink. JFM 829, 153 (2017) & JFM 829, 236 (2017) & JFM 836, 560 (2018)

dissipative anomaly (refined argument)

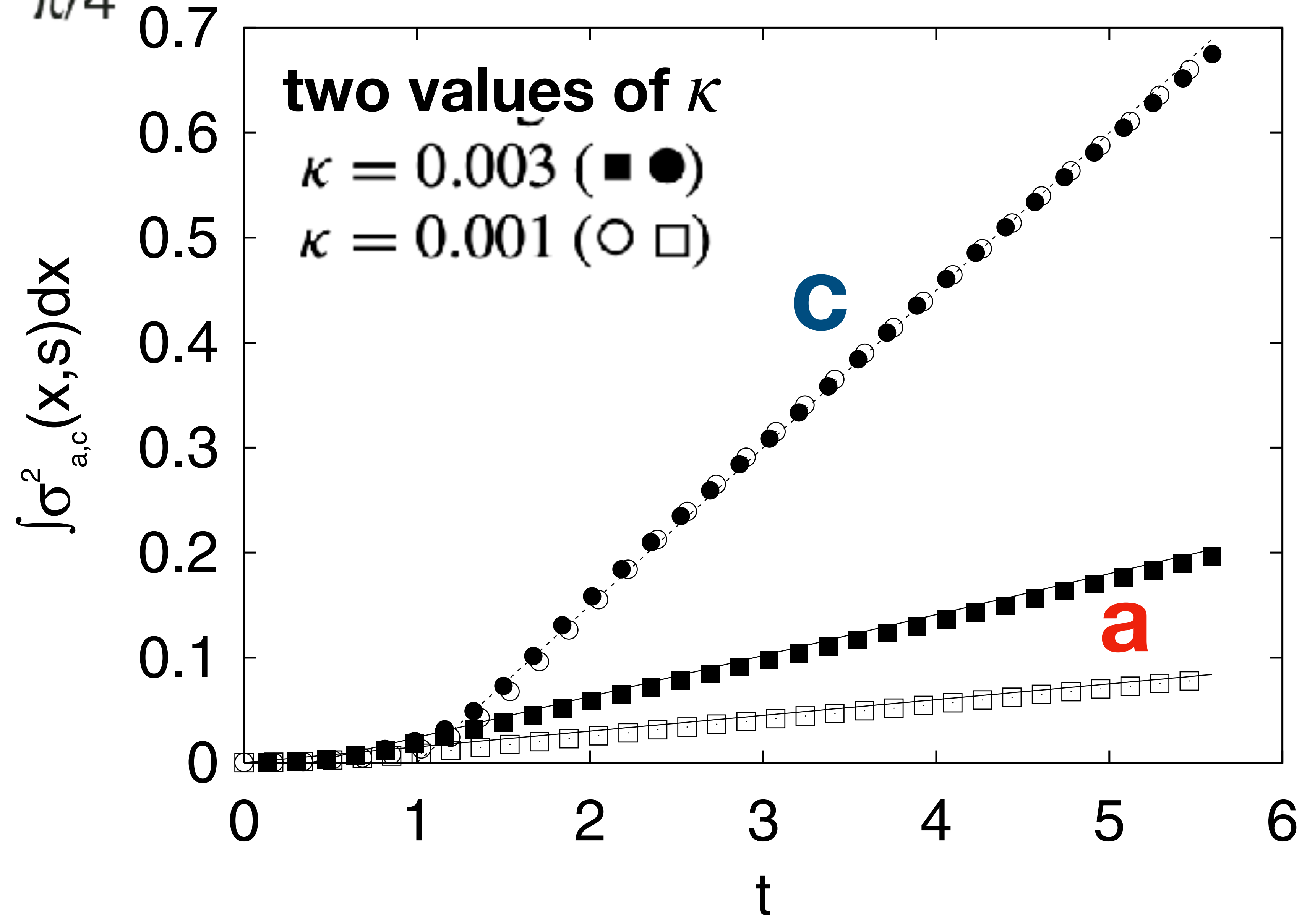


2D MHD + a passive scalar

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} - \Delta a \nabla a$$
$$\partial_t a + \boldsymbol{v} \cdot \nabla a = \kappa \Delta a + f_a$$
$$\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta c + f_c$$

the magnetic potential a
is an inverse cascading active scalar
↓
absence of dissipative anomaly

A Celani, M. C., A Mazzino, and M Vergassola.
New Journal of Physics 6, (2004): 72.



Lagrangian origin of Eulerian intermittency in passive scalars

Outline of the main steps

- First we show that moments of scalar increments are connected to N-point correlation functions
- Then we focus on the N=2 correlation function
- Then we consider the Kraichnan model and sketch how to derive an equation for the N-point correlation function
- Then we show that this equation leads us to the zero modes we saw for multiparticle dispersion
- Finally we show that they are responsible for anomalous scaling and universality

Multipoint correlation functions

$$C_N(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \langle \theta(\boldsymbol{x}_1, t)\theta(\boldsymbol{x}_2, t)\ldots\theta(\boldsymbol{x}_N, t)\rangle_{v,F}$$

Multipoint correlation functions

The goal is to understand the scaling of SF, which are connected to correlation functions:

$$C_N(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \langle \theta(\boldsymbol{x}_1, t)\theta(\boldsymbol{x}_2, t)\ldots\theta(\boldsymbol{x}_N, t)\rangle_{v,F}$$

$$S_N(r) = \langle(\theta(\boldsymbol{x}+\boldsymbol{r}, t) - \theta(\boldsymbol{x}, t))^N\rangle_{v,F} = \langle\left(\int_0^1 \partial_s\theta(x+sr)ds\right)^N\rangle_{v,F} = \int_0^1 ds_1 \ldots \int_0^1 ds_N \partial_{s_1}\ldots\partial_{s_N} C_N(\boldsymbol{x}+s_1\boldsymbol{r}, \ldots, \boldsymbol{x}+s_N\boldsymbol{r})$$

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In the following we assume Gaussian, uncorrelated forcing, acting at large scales

$$\langle F(\boldsymbol{x}, t)F(\boldsymbol{y}, t')\rangle_F = \delta(t-t')\Phi(|\boldsymbol{x}-\boldsymbol{y}|)$$

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We start with the 2-points Correlation function and use $\theta(\boldsymbol{x}, t) = \int^t ds \int d\boldsymbol{y} p_v(\boldsymbol{y}, s|\boldsymbol{x}, t)F(\boldsymbol{y}, s)$

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NB we could separate the averages on velocity and forcing because we are considering passive scalars

Multipoint correlation functions

The goal is to understand the scaling of SF, which are connected to correlation functions:

$$C_N(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \langle \theta(\boldsymbol{x}_1, t)\theta(\boldsymbol{x}_2, t) \ldots \theta(\boldsymbol{x}_N, t)\rangle_{v,F}$$

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$$\langle F(\boldsymbol{x}, t) F(\boldsymbol{y}, t') \rangle_F = \delta(t - t')\Phi(|\boldsymbol{x} - \boldsymbol{y}|)$$

We start with the 2-points Correlation function and use

$$\theta(\boldsymbol{x}, t) = \int^t ds \int d\boldsymbol{y} p_v(\boldsymbol{y}, s|\boldsymbol{x}, t) F(\boldsymbol{y}, s)$$

$$\begin{aligned}
C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \; & \langle \theta(\boldsymbol{x}_1, t)\theta(\boldsymbol{x}_2, t)\rangle_{F,v} = \int^t ds_1 \int^t ds_2 \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \langle p_v(\boldsymbol{y}_1, s_1|\boldsymbol{x}, t) p_v(\boldsymbol{y}_2, s_2|\boldsymbol{y}, t)\rangle_v \langle F(\boldsymbol{y}_1, s_1) F(\boldsymbol{y}_2, s_2)\rangle \\
= & \int^t ds_1 \int^t ds_2 \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \langle p_v(\boldsymbol{y}_1, s_1|\boldsymbol{x}_1, t) p_v(\boldsymbol{y}_2, s_2|\boldsymbol{x}_2, t)\rangle_v \Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)\delta(s_1 - s_2) \\
= & \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \langle p_v(\boldsymbol{y}_1, s|\boldsymbol{x}_1, t) p_v(\boldsymbol{y}_2, s|\boldsymbol{x}_2, t)\rangle_v \Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) \\
= & \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)
\end{aligned}$$

Multipoint correlation functions

The goal is to understand the scaling of SF, which are connected to correlation functions:

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$$= \int^t ds_1 \int^t ds_2 \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \langle p_v(\boldsymbol{y}_1, s_1|\boldsymbol{x}_1, t)p_v(\boldsymbol{y}_2, s_2|\boldsymbol{x}_2, t)\rangle_v \Phi(|\boldsymbol{y}_1-\boldsymbol{y}_2|)\delta(s_1-s_2)$$

$$= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \boxed{\langle p_v(\boldsymbol{y}_1, s|\boldsymbol{x}_1, t)p_v(\boldsymbol{y}_2, s|\boldsymbol{x}_2, t)\rangle_v} \Phi(|\boldsymbol{y}_1-\boldsymbol{y}_2|)$$

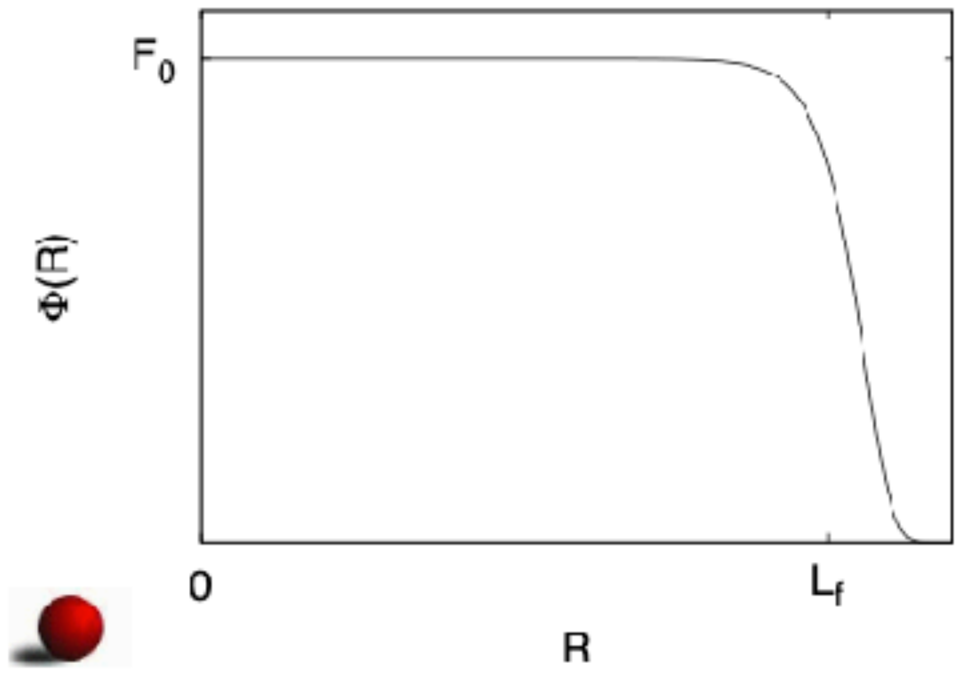
$$= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1-\boldsymbol{y}_2|)$$

Multipoint correlation functions

$$c(\boldsymbol{x},t) = \int_{-\infty}^{t} ds \int d\boldsymbol{y}\, p(\boldsymbol{y},s|\boldsymbol{x},t)\, F_c(\boldsymbol{y},s)$$

$$\langle F_c(y1)F_c(y2)\rangle = \Phi(R)\delta(s_1 - s_2) \qquad y1 = (\boldsymbol{y}_1, s_1) \quad x1 = (\boldsymbol{x}_1, t_1)$$

$$C(r) = \langle c(x1)c(x2)\rangle_{v,F} = \iint_{-\infty}^{t} ds_1 ds_2 \iint d\boldsymbol{y}_1 d\boldsymbol{y}_2\, \underbrace{\langle p(y1|x1)p(y2|x2)\rangle_v}_{P_2(\boldsymbol{R},s|\boldsymbol{r},t)}\, \underbrace{\langle F_c(y1)F_c(y2)\rangle_F}_{\Phi(R)\delta(s_1-s_2)}$$



$$C(r) = \int_{-\infty}^{t} ds \int d\boldsymbol{R}\, P_2(\boldsymbol{R},s|\boldsymbol{r},t)\, \Phi(R) \approx \Phi(0)\, \mathcal{T}(r, L_f)$$

$$S(r) = 2(C(0) - C(r)) \approx \Phi(0)\left[\mathcal{T}(0, L_f) - \mathcal{T}(r, L_f)\right]$$

$$\mathcal{T}(r, L_f) = \text{time spent by a pair to go from distance } r \text{ to } L_f$$

RICHARDSON DISPERSION

$$dR/dt = \delta_R v \approx R^{1/3}$$

$$R(t) \approx (R(0)^{2/3} + t)^{3/2} \to t^{3/2}$$

$R(t) \sim t^{3/2} \Longrightarrow \mathcal{T}(0, L_f) - \mathcal{T}(r, L_f) \sim r^{2/3}$

We find again $S(r) \sim r^{2/3}$

Velocity roughness $\Longrightarrow$ non-uniqueness of Lagrangian paths

2-points correlation function

(Kraichnan)

$$\partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \tag{14}$$

with $P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \delta(\boldsymbol{y}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{y}_2 - \boldsymbol{x}_2)$ and $\mathcal{M}_2$ acting on $\boldsymbol{x}_1, \boldsymbol{x}_2$. $\mathcal{M}_2$ **a given operator**

$$\partial_t C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \overset{\bigstar}{\partial_t} \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)$$

★ Remember that $\partial_t \int^t f(s,t)ds = \int^t \partial_t f(s,t)ds + f(t,t)$.

2-points correlation function

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with $P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \delta(\boldsymbol{y}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{y}_2 - \boldsymbol{x}_2)$ and $\mathcal{M}_2$ acting on $\boldsymbol{x}_1, \boldsymbol{x}_2$. $\mathcal{M}_2$ **a given operator**

$$\begin{aligned}
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&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)
\end{aligned}$$

★ Remember that $\partial_t \int^t f(s,t)ds = \int^t \partial_t f(s,t)ds + f(t,t)$.

2-points correlation function

(Kraichnan)

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&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)
\end{aligned}$$

$\bigstar$ Remember that $\partial_t \int^t f(s,t)ds = \int^t \partial_t f(s,t)ds + f(t,t)$.

2-points correlation function

(Kraichnan)

$$\partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \tag{14}$$

with $P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \delta(\boldsymbol{y}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{y}_2 - \boldsymbol{x}_2)$ and $\mathcal{M}_2$ acting on $\boldsymbol{x}_1, \boldsymbol{x}_2$. $\mathcal{M}_2$ **a given operator**

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&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) \\
&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) \\
&= \mathcal{M}_2 \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)
\end{aligned}$$

$\bigstar$ Remember that $\partial_t \int^t f(s,t) ds = \int^t \partial_t f(s,t) ds + f(t,t)$.

2-points correlation function

(Kraichnan)

$$\partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \tag{14}$$

with $P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \delta(\boldsymbol{y}_1 - \boldsymbol{x}_1)\delta(\boldsymbol{y}_2 - \boldsymbol{x}_2)$ and $\mathcal{M}_2$ acting on $\boldsymbol{x}_1, \boldsymbol{x}_2$. $\mathcal{M}_2$ **a given operator**

$$\begin{aligned}
\partial_t C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) &= \overset{\bigstar}{\partial_t} \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) \\
&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \partial_t P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; t|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) \\
&= \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \mathcal{M}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) \\
&= \mathcal{M}_2 \int^t ds \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t)\Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) \\
&= \mathcal{M}_2 C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)
\end{aligned}$$

★ Remember that $\partial_t \int^t f(s,t)ds = \int^t \partial_t f(s,t)ds + f(t,t)$.

Multipoint correlation functions

K. Gawedzki "Soluble models of turbulent advection" arXiv preprint nlin/0207058 (2002).

A simpler derivation

Assume to know $C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; s)$ what is its value at time t?

$$C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) C_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s) + \int_s^t d\tau \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)$$

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$$\begin{aligned} \partial_t C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) &= \mathcal{M}_2 \left[\int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) C_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s) + \int_s^t d\tau \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|) \right] \\ &+ \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) = \mathcal{M}_2 C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) + \Phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) \end{aligned}$$

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Now let's do a step forward: consider $C_N(\underline{\boldsymbol{x}}; t) = \langle \theta(\boldsymbol{x}_1, t) \dots \theta(\boldsymbol{x}_N, t)\rangle_{v,F}$ with $\underline{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$. Generalizing the result for C_2 we can write:

$$C_N(\underline{\boldsymbol{x}}; t) = \int d\underline{\boldsymbol{y}} P_N(\underline{\boldsymbol{y}}; s|\underline{\boldsymbol{x}}; t) C_N(\underline{\boldsymbol{y}}; s) + \int_s^t d\tau \int d\underline{\boldsymbol{y}} P_N(\underline{\boldsymbol{y}}; s|\underline{\boldsymbol{x}}; t)(C_{N-2} \otimes \Phi)(\underline{\boldsymbol{y}}, \tau)$$

where

$$(C_{N-2} \otimes \Phi)(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N, \tau) = \sum_{n<m} C_{N-2}(\boldsymbol{x}_1, \dots [\hat{n}] \dots [\hat{m}] \dots, \boldsymbol{x}_N)\Phi(|\boldsymbol{x}_n - \boldsymbol{x}_m|)$$

Multipoint correlation functions

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Assume to know $C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; s)$ what is its value at time t?

$$C_2(\boldsymbol{x}_1, \boldsymbol{x}_2; t) = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) C_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s) + \int_s^t d\tau \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 P_2(\boldsymbol{y}_1, \boldsymbol{y}_2; s|\boldsymbol{x}_1, \boldsymbol{x}_2; t) \Phi(|\boldsymbol{y}_1 - \boldsymbol{y}_2|)$$

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where

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As for $N = 2$ doing the time derivative one obtains

$$\partial_t C_N = \mathcal{M}_N C_N + (C_{N-2} \otimes \Phi)$$

Kraichnan model

$$P_N = P_N(\underline{\boldsymbol{r}}; s|\underline{\boldsymbol{R}}; t) \quad \Longrightarrow \quad \partial_t P_N = \mathcal{M}_N P_N$$

$$C_N(\underline{\boldsymbol{R}}; t) \quad \Longrightarrow \quad \partial_t C_N = \mathcal{M}_N C_N + C_{N-2} \otimes \Phi$$

As we said this require special properties of the velocity field: Gaussianity and time-uncorrelation

$$\langle v_\alpha(\boldsymbol{x}, t) v_\beta(\boldsymbol{x}+\boldsymbol{r}, t')\rangle_v = \delta(t-t') D_{\alpha,\beta}(\boldsymbol{x}-\boldsymbol{y})$$

incompressibility→ $\partial_\alpha D_{\alpha,\beta} = 0$

(in the inertial range) $\eta \ll r \ll L_v$

$$d_{\alpha,\beta}(\boldsymbol{r}) = D_1 r^\xi \left((d-1+\xi)\delta_{\alpha\beta} - \xi \frac{r_\alpha r_\beta}{r^2}\right)$$

$$\mathcal{M}_N = -\sum_{n<m} d_{\alpha\beta}(\boldsymbol{r}_n - \boldsymbol{r}_m)\partial_{r_{n,\alpha}}\partial_{r_{m,\beta}} + \kappa \sum_{n=1}^{N} \Delta_{r_n} \cancel{+ D_0 \delta_{\alpha\beta} \left(\sum_{n=1}^{N} \partial_{r_{n,\alpha}}\right)^2}$$

not apply to space invariant functions

Dimensional analysis

$$d_{\alpha,\beta}(\boldsymbol{r}) = D_1 r^{\xi}\left((d-1+\xi)\delta_{\alpha\beta} - \xi\frac{r_\alpha r_\beta}{r^2}\right)$$

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$$\partial_t C_N = \mathcal{M}_N C_N + C_{N-2}\otimes\Phi$$

at stationarity

$$-\mathcal{M}_N C_N = C_{N-2}\otimes\Phi$$

Dimensional analysis

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$$[\mathcal{M}_N] = L^{\xi-2}$$

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Dimensional analysis

$$[\mathcal{M}_N] = L^{\xi-2}$$

time $T \sim [M_N]^{-1} \sim L^{2-\xi}$ indeed $\xi = 0$ $T \sim L^2$ as we recover normal diffusion

$$\partial_t C_N = \mathcal{M}_N C_N + C_{N-2} \otimes \Phi$$

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For small r

$$\mathcal{M}_2 C_2 \sim \Phi(0) \Longrightarrow [C_2] = [\mathcal{M}_2]^{-1} = L^{2-\xi} \sim C(r) \sim r^{2-\xi}$$

Dimensional analysis

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Dimensional analysis

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For small r

$$\mathcal{M}_2 C_2 \sim \Phi(0) \implies [C_2] = [\mathcal{M}_2]^{-1} = L^{2-\xi} \sim C(r) \sim r^{2-\xi}$$

$$\mathcal{M}_4 C_4 \sim \Phi(0)C_2 \implies [C_4] = [\mathcal{M}_4]^{-1}[C_2] = L^{2(2-\xi)}$$

Dimensional analysis

$$d_{\alpha,\beta}(\boldsymbol{r}) = D_1 r^\xi \left((d-1+\xi)\delta_{\alpha\beta} - \xi \frac{r_\alpha r_\beta}{r^2}\right)$$

$$\mathcal{M}_N = -\sum_{n<m} d_{\alpha\beta}(\boldsymbol{r}_n - \boldsymbol{r}_m)\partial_{r_{n,\alpha}}\partial_{r_{m,\beta}} + \kappa \sum_{n=1}^{N} \Delta_{r_n}$$

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For small r

$$\mathcal{M}_2 C_2 \sim \Phi(0) \Longrightarrow [C_2] = [\mathcal{M}_2]^{-1} = L^{2-\xi} \sim C(r) \sim r^{2-\xi}$$

$$\mathcal{M}_4 C_4 \sim \Phi(0) C_2 \Longrightarrow [C_4] = [\mathcal{M}_4]^{-1}[C_2] = L^{2(2-\xi)}$$

$$\vdots$$

$$\mathcal{M}_N C_N \sim \Phi(0) C_{N-2} \Longrightarrow [C_N] = [\mathcal{M}_N]^{-1}[C_{N-2}] = L^{N(2-\xi)}$$

$$C_N \sim L^{\zeta_N^{dim} = N(2-\xi)}$$

Where does it come the anomalous scaling?

Dominance of Zero Modes

$$-\mathcal{M}_N C_N = C_{N-2} \otimes \Phi$$

The most general solution is given by

$$\mathcal{M}_N Z_N = 0$$

$$C_N = \underbrace{\mathcal{M}_N^{-1} C_{N-2} \otimes \Phi}_{L^{\zeta_N^{dim}}} + \underbrace{Z_N}_{L^{\zeta_N^{a}}}$$

$$\boxed{\zeta_N^a < \zeta_N^{dim} = N(2-\xi)}$$

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$$\boxed{\zeta_N^a < \zeta_N^{dim} = N(2-\xi)}$$

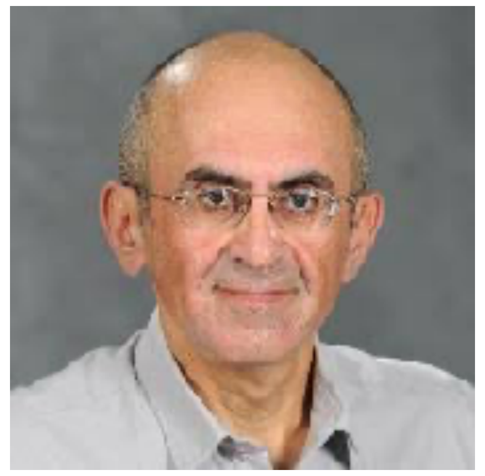
1995

Gawdezki & Kupianen $\xi \to 0$ Perturbation around Brownian motion

$$\zeta_{N,0} = \frac{N}{2}(2-\xi) - \frac{N(N-2)}{2(d+2)}\xi + \mathcal{O}(\xi^2),$$

Chertkov, Falkovich & Lebedev $d \to \infty$

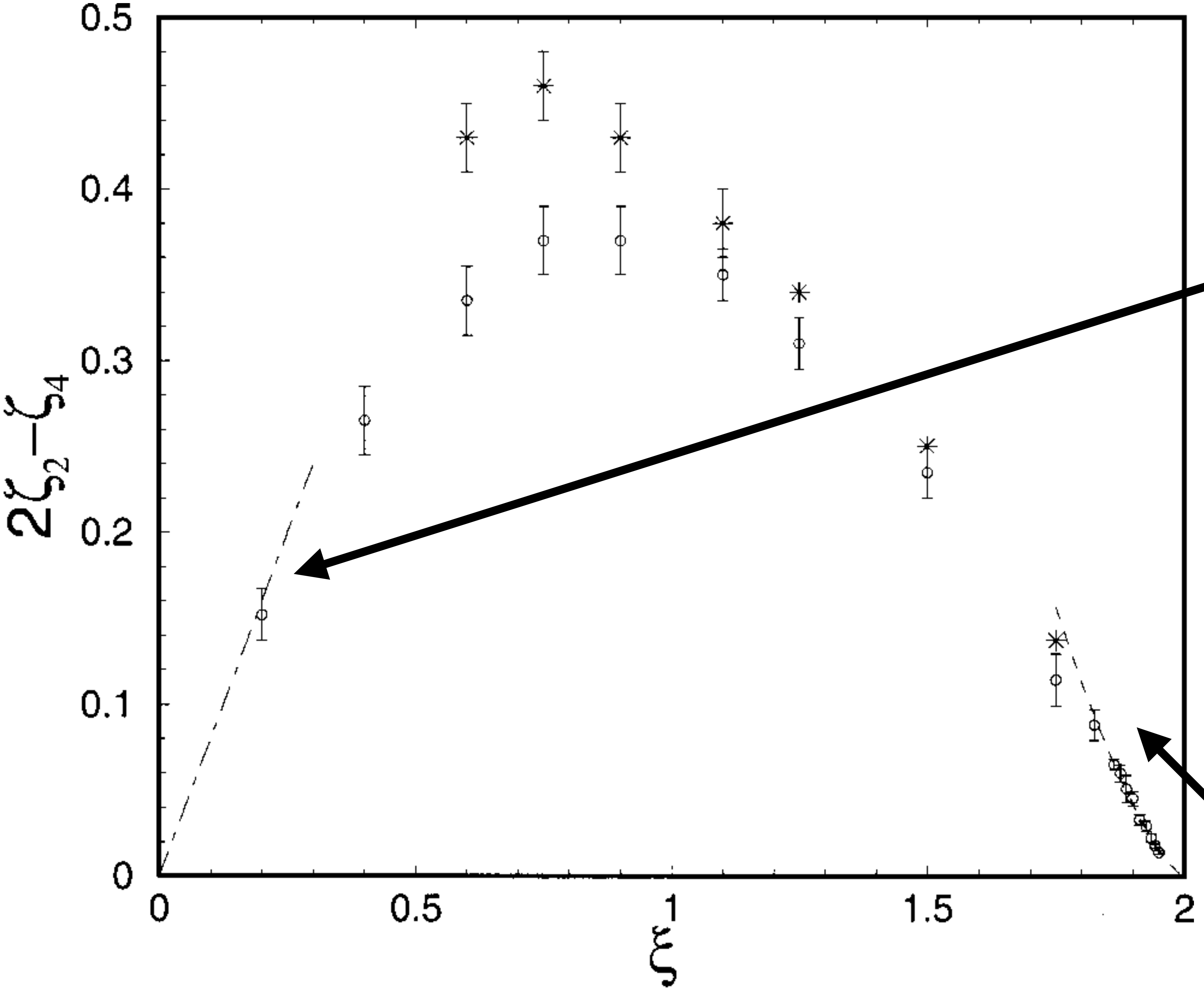
$$\zeta_{N,0} = \frac{N}{2}(2-\xi) - \frac{N(N-2)}{2d}\xi + \mathcal{O}\left(\frac{1}{d^2}\right)$$



Shraiman & Siggia $\xi \to 2$ Perturbation around Batchelor limit

also found anomalous exponents for zero modes

Dominance of Zero Modes



Frisch, Mazzino, Vergassola 1998

1995

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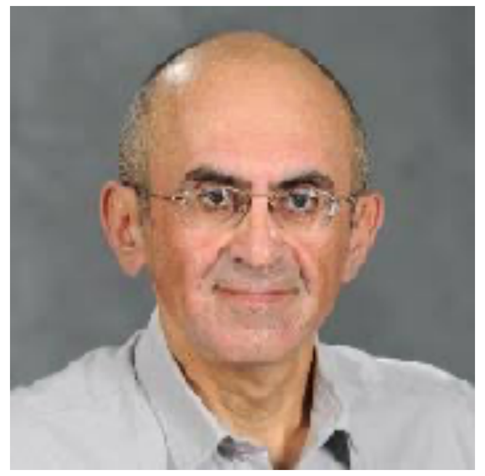


Shraiman & Siggia $\xi \to 2$ Perturbation around Batchelor limit

also found anomalous exponents for zero modes

Physical (Lagrangian) interpretation of zero modes

Kraichnan model

$$\langle v_\alpha(\boldsymbol{x},t)v_\beta(\boldsymbol{x}+\boldsymbol{r},t')\rangle_v = \delta(t-t')D_{\alpha,\beta}(\boldsymbol{x}-\boldsymbol{y})$$

Time reversible

Lagrangian trajectories are reversible

Backward and Forward probabilities coincide

For any test function $f(\underline{\boldsymbol{x}})$ of N points we can define the Lagrangian averages

$$\langle f\rangle(t;\underline{\boldsymbol{x}}_0) = \int d\underline{\boldsymbol{x}} P_N(\underline{\boldsymbol{x}},t|\underline{\boldsymbol{x}}_0,0)f(\underline{\boldsymbol{x}})$$

If f is a scaling function $f(\lambda\underline{\boldsymbol{x}}) = \lambda^\sigma f(\underline{\boldsymbol{x}})$ **we expect** $\langle f\rangle(t;\underline{\boldsymbol{x}}_0) \sim t^{\sigma/(2-\xi)}$

e.g. Relative dispersion $f(\boldsymbol{x}_1,\boldsymbol{x}_2) = ||\boldsymbol{x}_1-\boldsymbol{x}_2||^2 = R^2$ $\quad\langle R^2\rangle \sim t^{2/(2-\xi)}$

How does behave the Lagrangian average of zero modes (which are scaling functions of N-points)

$$\langle Z_N\rangle(t;\underline{\boldsymbol{x}}_0) = \int d\underline{\boldsymbol{x}} P_N(\underline{\boldsymbol{x}},t|\underline{\boldsymbol{x}}_0,0)Z_N(\underline{\boldsymbol{x}})$$

$$\frac{d}{dt}\langle Z_N\rangle(t;\underline{\boldsymbol{x}}_0) = \int d\underline{\boldsymbol{x}}\partial_t P_N(\underline{\boldsymbol{x}},t|\underline{\boldsymbol{x}}_0,0)Z_N(\underline{\boldsymbol{x}}) = \langle\mathcal{M}_N Z_N\rangle(t|0,\underline{\boldsymbol{x}}_0) = 0!$$

Zero modes are statistically conserved by the Lagrangian flow

e.g. $\langle R_{12}^2 - R_{34}^2\rangle = \langle R_{12}^2\rangle - \langle R_{34}^2\rangle = (R_{12}^2(0)+2Dt) - (R_{34}^2(0)+2Dt) = const$

($\xi\to 0 \quad \mathscr{M}_N \to \Delta_N \quad Z_N \to$ **harmonic polynomials**

For non-Kraichnan flows?

Celani & Vergassola 2001

passive scalar with a gradient

$$\Theta(\boldsymbol{x},t) = \theta(\boldsymbol{x},t) + \boldsymbol{g}\cdot\boldsymbol{x}$$

$$\partial_t\theta(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\cdot\nabla\,\theta(\boldsymbol{r},t) = \kappa\Delta\theta(\boldsymbol{r},t) - \boldsymbol{g}\cdot\boldsymbol{v}$$

velocity NS-2D in inverse cascade

with Gaussian forcing $C_3 = 0$

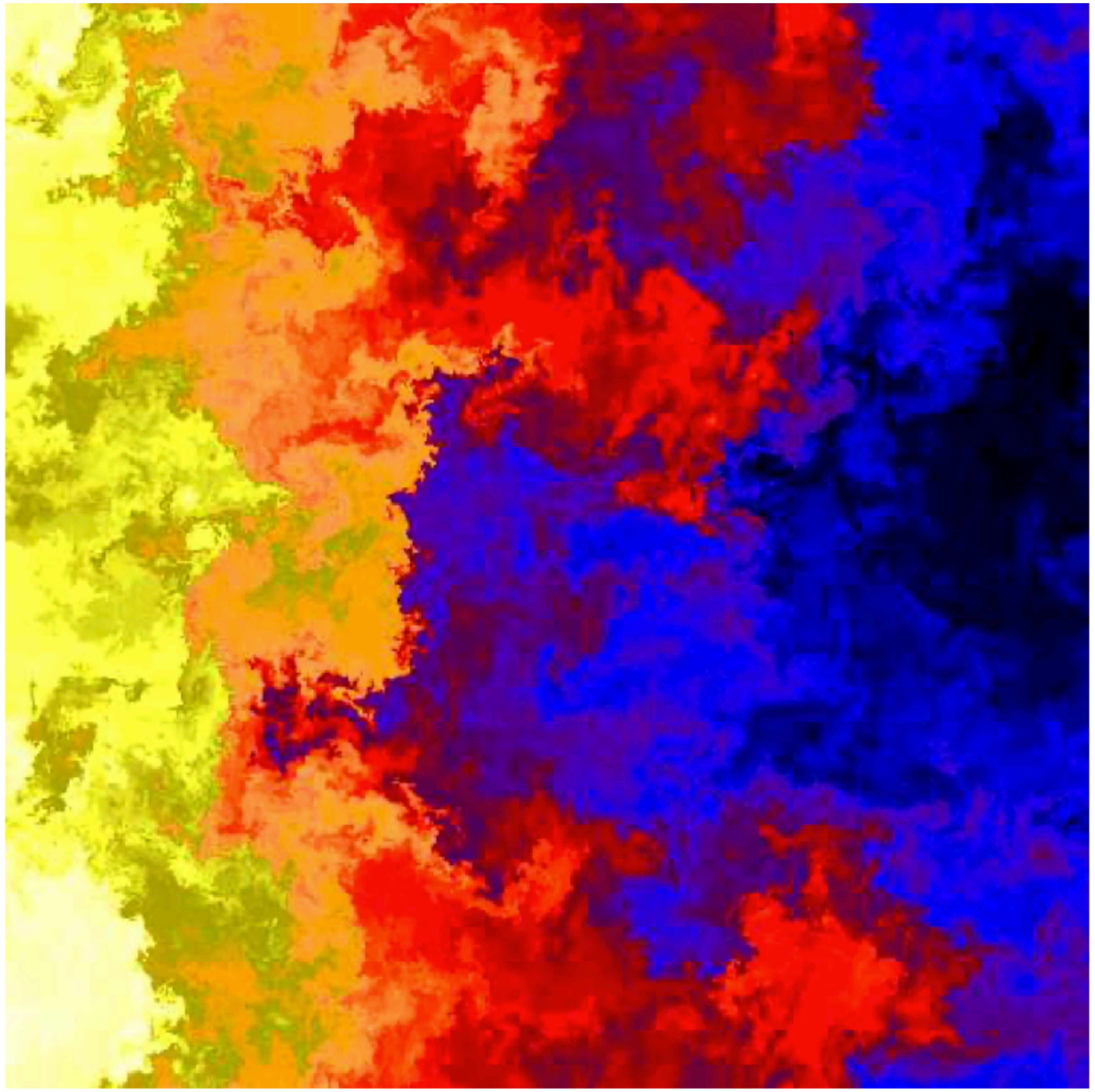
with the gradient $C_3 \neq 0$

$$S_{2N}^{gauss}(r) \sim S_{2N}^{grad}(r)$$

$$S_{2N+1}^{gauss}(r) = 0 \quad S_{2N+1}^{grad}(r) \neq 0$$

by dimensional arguments

$$S_{2N+1}^{grad}(r) \sim (\boldsymbol{g}\cdot\boldsymbol{r})S_{2N}^{grad}$$



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$$\partial_t \theta(\boldsymbol{r}, t) + \boldsymbol{v}(\boldsymbol{r}, t) \cdot \nabla\, \theta(\boldsymbol{r}, t) = \kappa \Delta \theta(\boldsymbol{r}, t) - \boldsymbol{g} \cdot \boldsymbol{v}$$

velocity NS-2D in inverse cascade

with Gaussian forcing $C_3 = 0$

with the gradient $C_3 \neq 0$

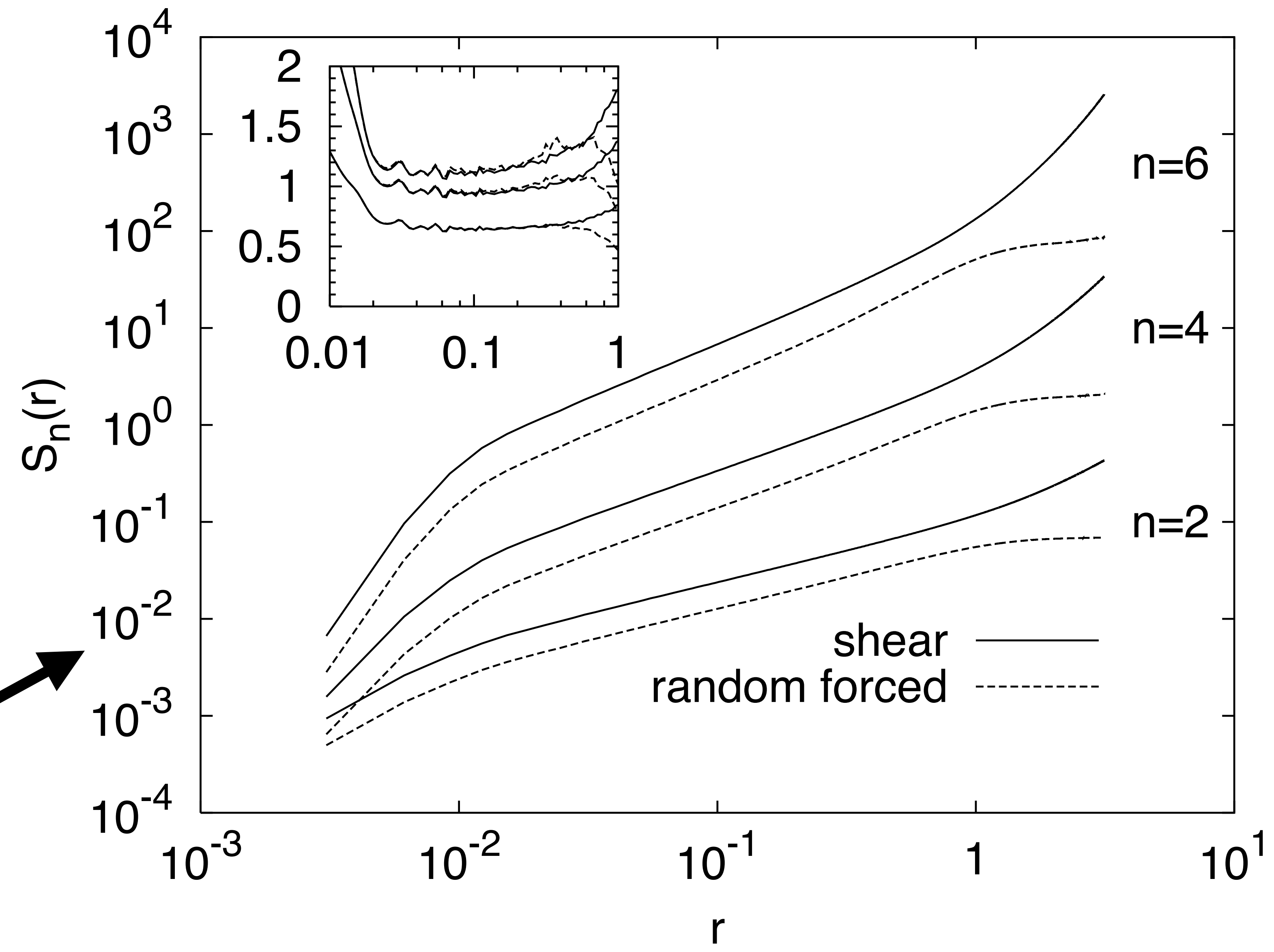
$$S^{gauss}_{2N}(r) \sim S^{grad}_{2N}(r)$$

universality w.r.t. forcing

$$S^{gauss}_{2N+1}(r) = 0 \quad S^{grad}_{2N+1}(r) \neq 0$$

by dimensional arguments

$$S^{grad}_{2N+1}(r) \sim (\boldsymbol{g} \cdot \boldsymbol{r}) S^{grad}_{2N}$$

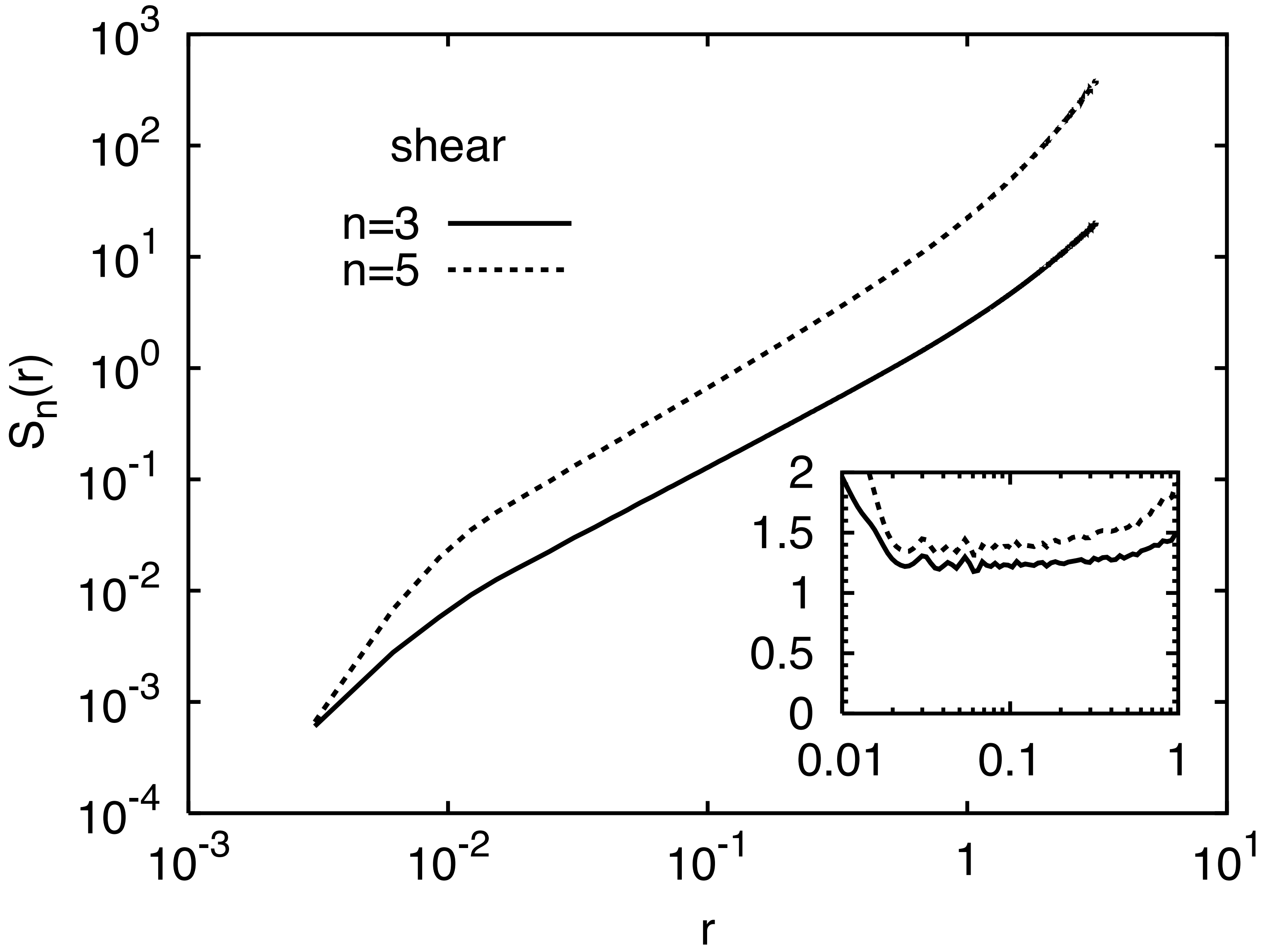


scaling is anomalous

$$\zeta_2 = 0.66 \pm 0.03,\ \zeta_4 = 0.95 \pm 0.04$$

$$\zeta_6 = 1.11 \pm 0.04$$

dim: 2n/3



scaling is anomalous

$$\zeta_3 = 1.25 \pm 0.04,\ \zeta_5 = 1.38 \pm 0.07$$

dim: 5/3 7/3

For non-Kraichnan flows?

Celani & Vergassola 2001

passive scalar with a gradient

$$\Theta(\boldsymbol{x},t) = \theta(\boldsymbol{x},t) + \boldsymbol{g}\cdot\boldsymbol{x}$$

$$\partial_t \theta(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\cdot\nabla\,\theta(\boldsymbol{r},t) = \kappa\Delta\theta(\boldsymbol{r},t) - \boldsymbol{g}\cdot\boldsymbol{v}$$

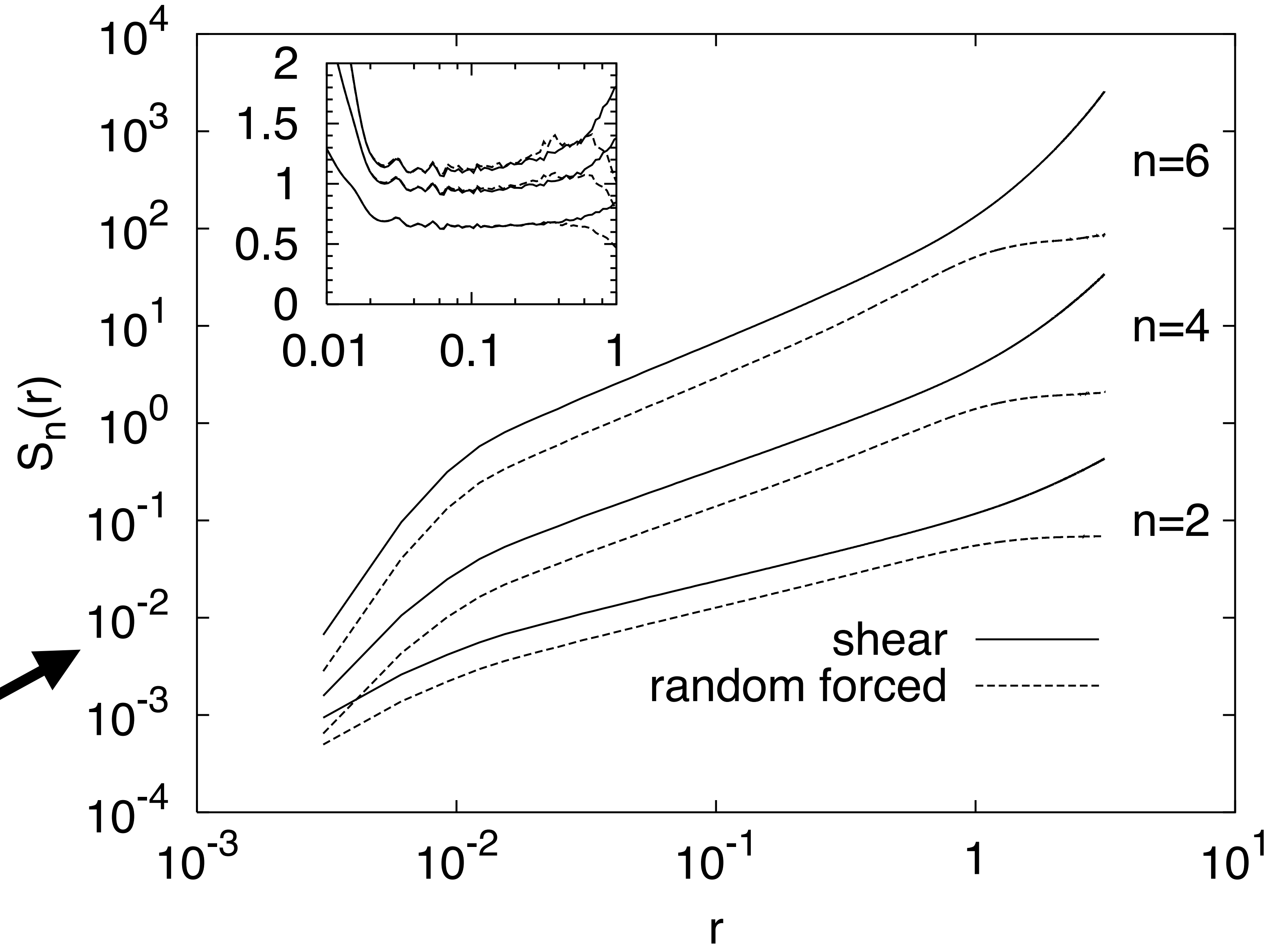
velocity NS-2D in inverse cascade

with Gaussian forcing $C_3 = 0$

with the gradient $C_3 \neq 0$

$$S^{gauss}_{2N}(r) \sim S^{grad}_{2N}(r)$$

universality w.r.t. forcing



scaling is anomalous

$$\zeta_2 = 0.66 \pm 0.03,\ \zeta_4 = 0.95 \pm 0.04$$

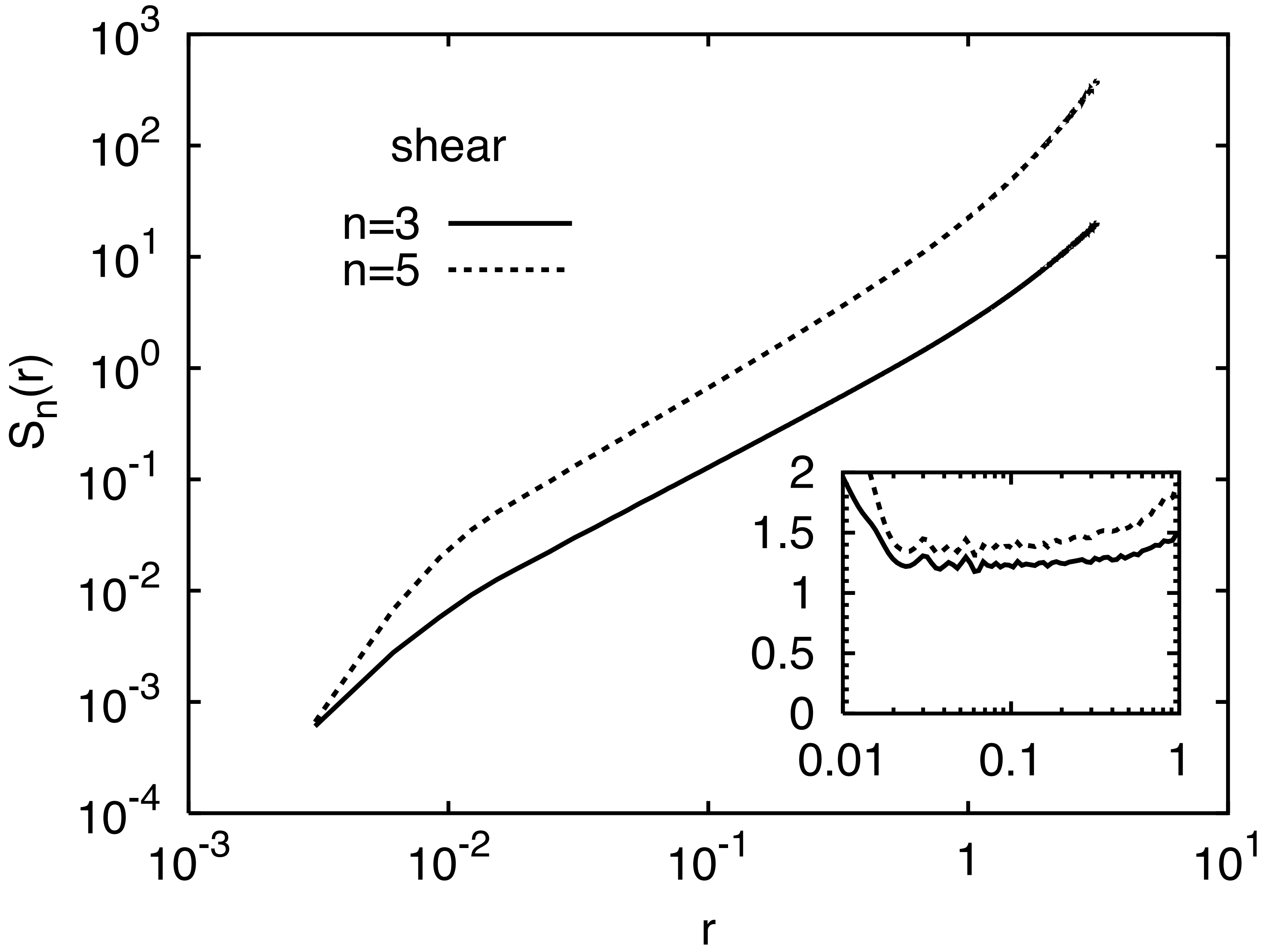
$$\zeta_6 = 1.11 \pm 0.04$$

dim: 2n/3

$$S^{gauss}_{2N+1}(r) = 0 \quad S^{grad}_{2N+1}(r) \neq 0$$

by dimensional arguments

$$S^{grad}_{2N+1}(r) \sim (\boldsymbol{g}\cdot\boldsymbol{r}) S^{grad}_{2N}$$



scaling is anomalous

$$\zeta_3 = 1.25 \pm 0.04,\ \zeta_5 = 1.38 \pm 0.07$$

dim: 5/3 7/3

DOES IT HOLD THE ZERO MODES PICTURE HERE?

3-points correlation function

$$C_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$$

size $R^2 = (r_{12}^2 + r_{23}^2 + r_{31}^2)/3$

shape $\chi = 1/2 \tan^{-1}\left[\dfrac{2\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2}{(\boldsymbol{\rho}_1^2 - \boldsymbol{\rho}_2^2)}\right]; \quad w = 2\,\dfrac{|\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|}{R}$

ϕ **orientation of the triangle**

$$\boldsymbol{\rho}_1 = (\boldsymbol{r}_1 - \boldsymbol{r}_2)/\sqrt{2}$$

$$\boldsymbol{\rho}_2 = (\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3)/\sqrt{6}$$

$$C_3(\underline{\boldsymbol{r}}) = R^{\zeta_3} f(\chi, w) \cos\varphi + \ldots$$

3-points correlation function

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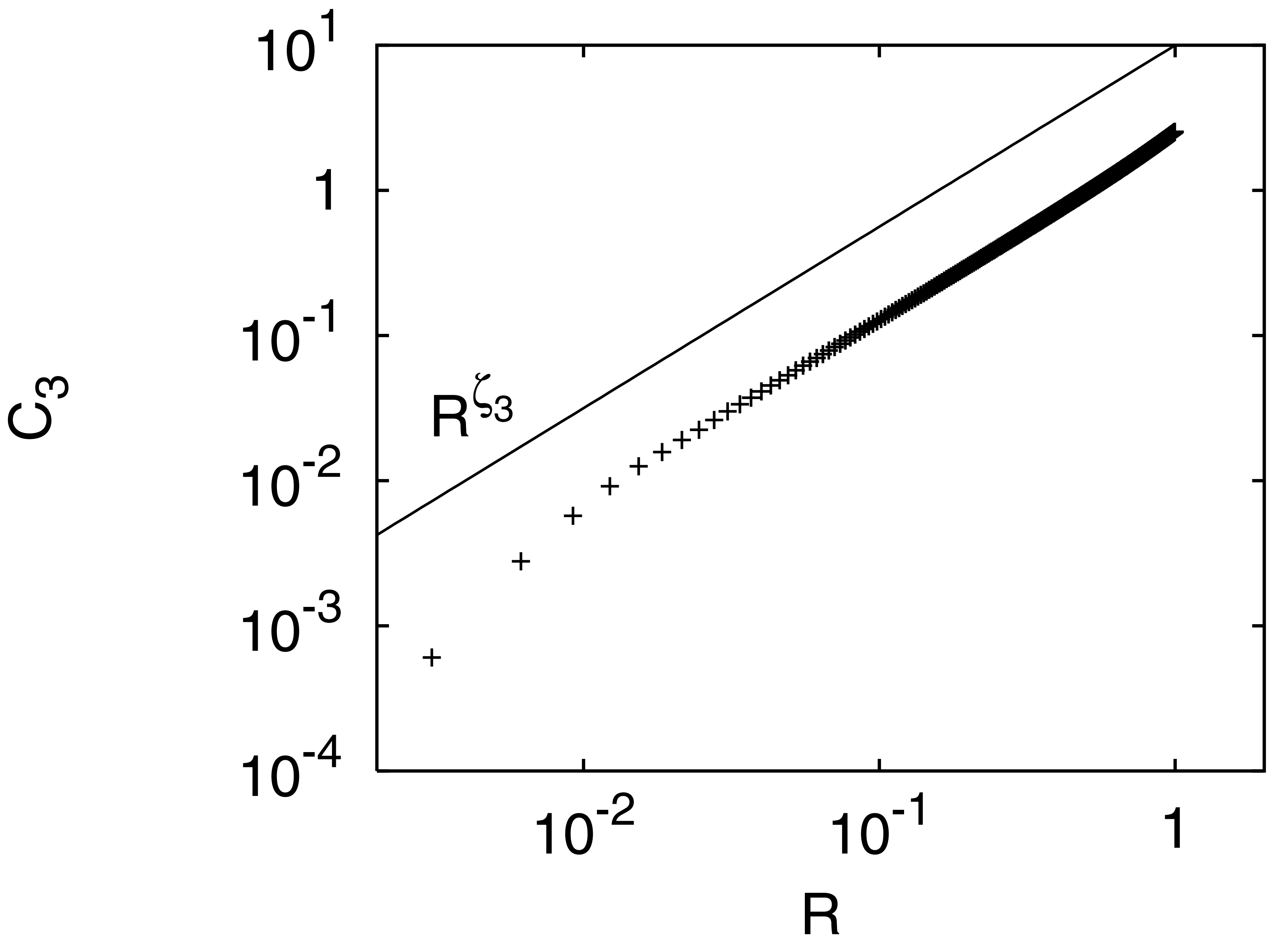
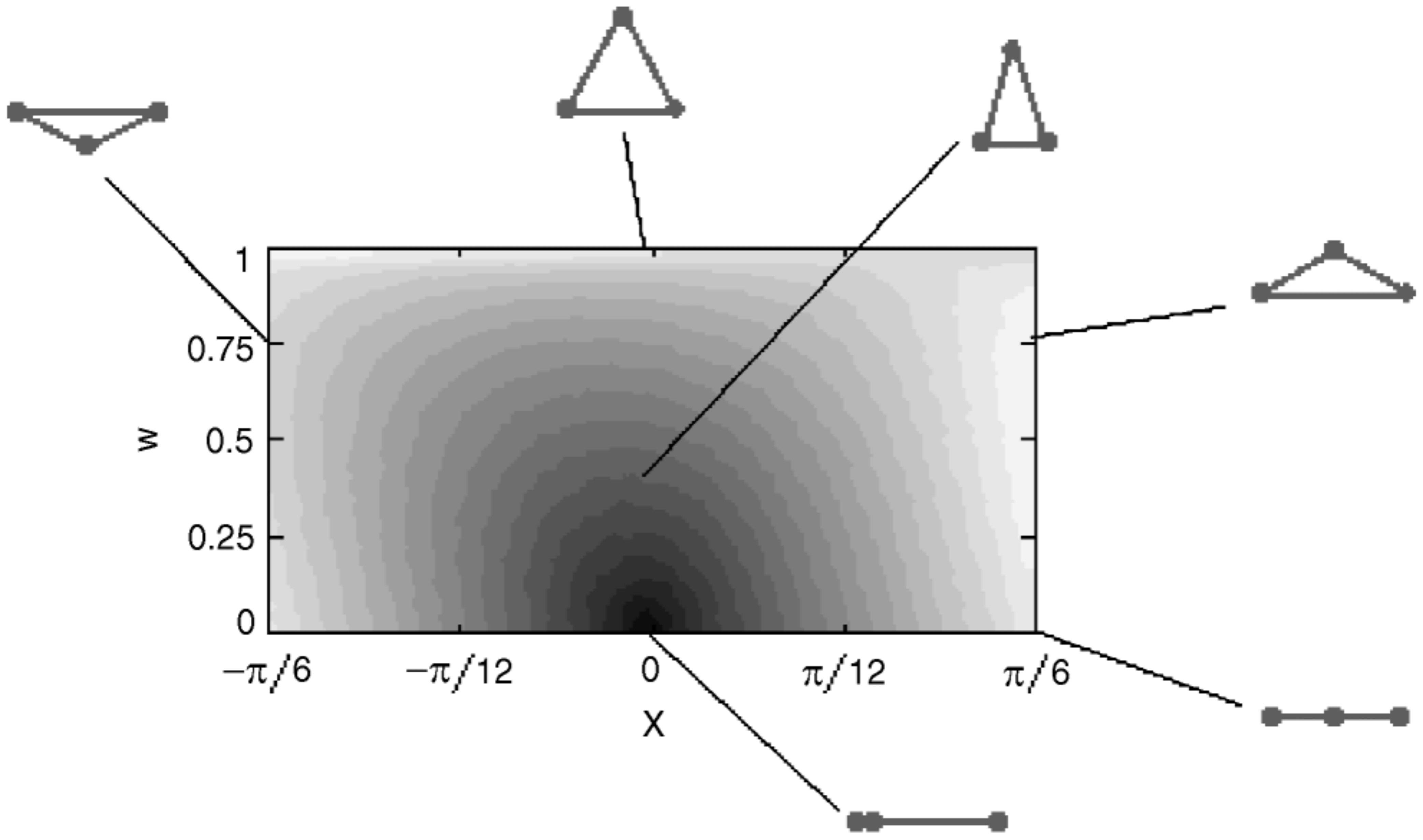


$$C_3(\underline{\boldsymbol{r}}) = R^{\zeta_3} f(\chi, w) \cos\varphi + \ldots$$

$$1.25 = \zeta_3 < \zeta_3^{dim} = 5/3$$

$$C_3(\underline{\boldsymbol{r}}) = Z_3(\underline{\boldsymbol{r}}) + subdominant$$ **?**



YES: 3-points correlation is statistically preserved

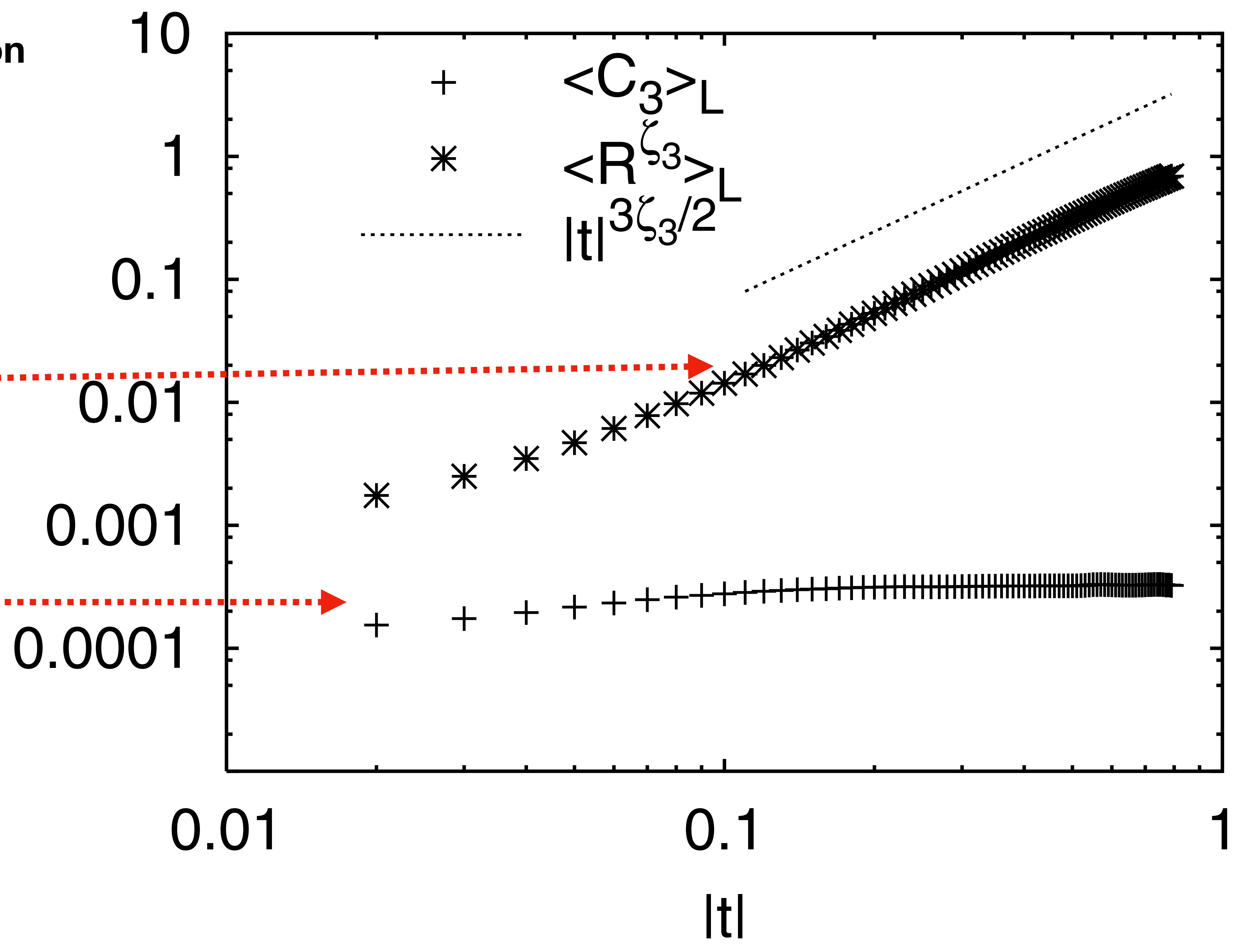
Lagrangian average of the 3-points correlation function

$$\langle f \rangle (t; \underline{\boldsymbol{x}}_0) = \int d\underline{\boldsymbol{x}} P_N(\underline{\boldsymbol{x}}, t | \underline{\boldsymbol{x}}_0, 0) f(\underline{\boldsymbol{x}})$$

$$f = R^2 = (r_{12}^2 + r_{23}^2 + r_{31}^2)/3$$

The size growth is compensated by the shape evolution

$$f = C_3(\underline{\boldsymbol{r}})$$



A side observation: saturation of intermittency

$$S_n(r) = \langle(\delta_r c)^n\rangle$$

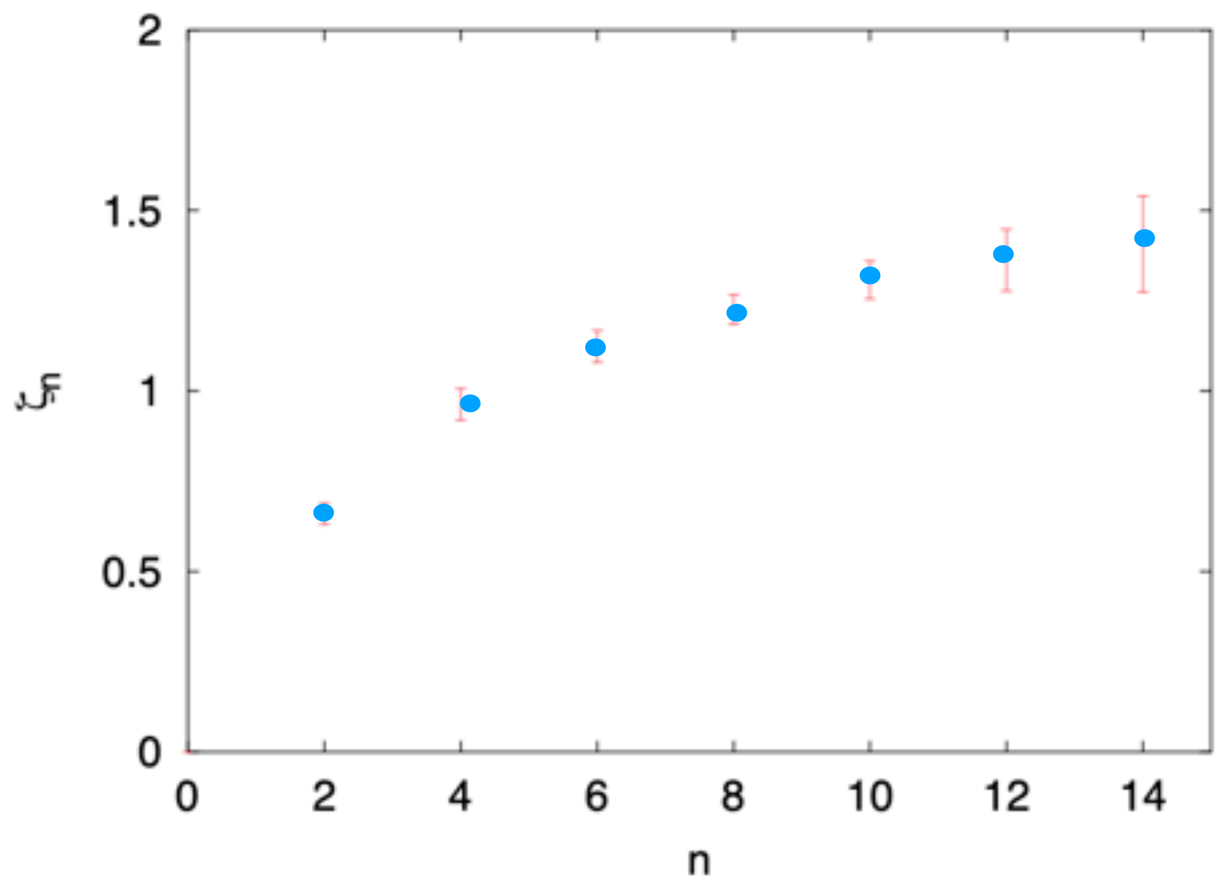
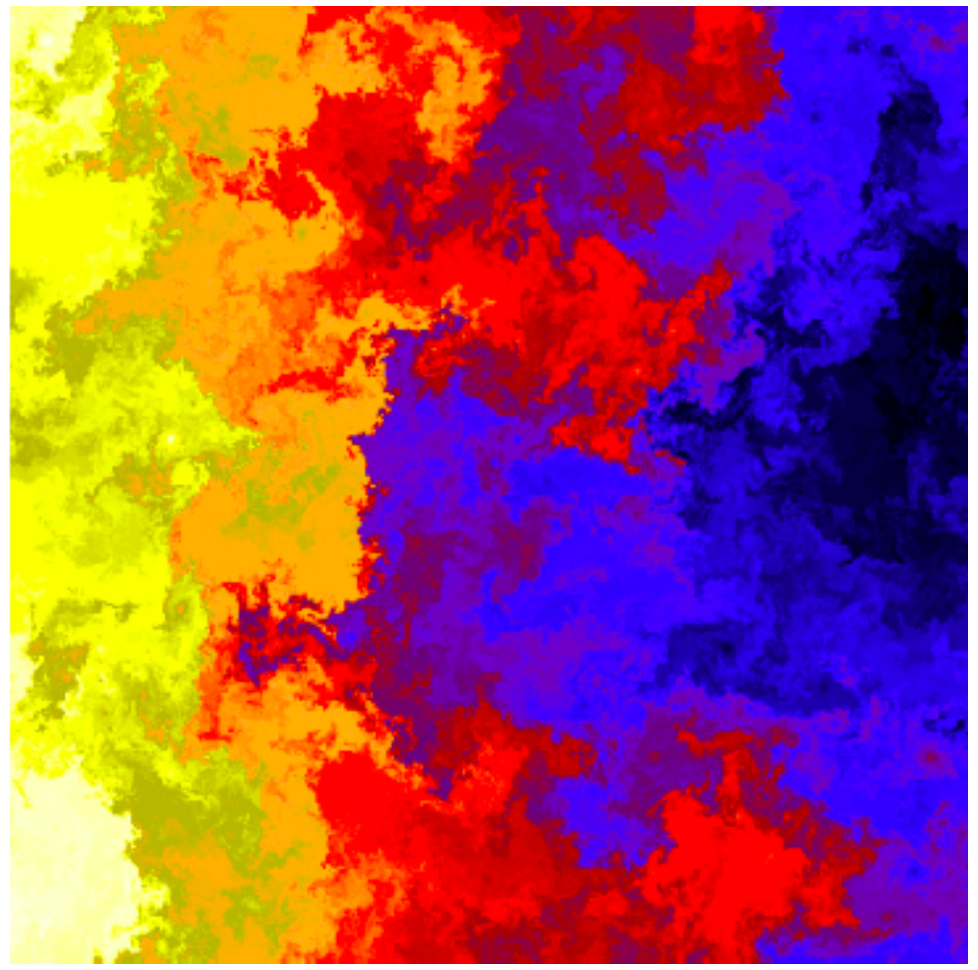


FIG. 9. Measured scaling exponent ζ_n for the Navier–Stokes advection. Error bars are estimated by the rms fluctuations of local scaling exponents.



Celani & Vergassola, Phys. Rev. Lett. **86**, 424 (2001)

The cliffs observed in scalar fields are strikingly suggestive of quasi-discontinuities. When smaller and smaller molecular diffusivities are considered, the minimal width of the fronts shrinks with the dissipation scale, with their maximum amplitude remaining comparable to the scalar rms value. Simple phenomenology suggests that the presence of such structures, corresponding to a local Hölder exponent equal to zero, might induce a vanishing slope in the structure function scaling exponent curve. The fronts being the strongest possible events, this behavior should take place for large enough orders, whence the possible saturation $\zeta_n \to$ const for high n's.

A. Celani, A Lanotte, A Mazzino, M Vergassola PRL 84, 2385 (2000) Pof 13 1768 (2001)

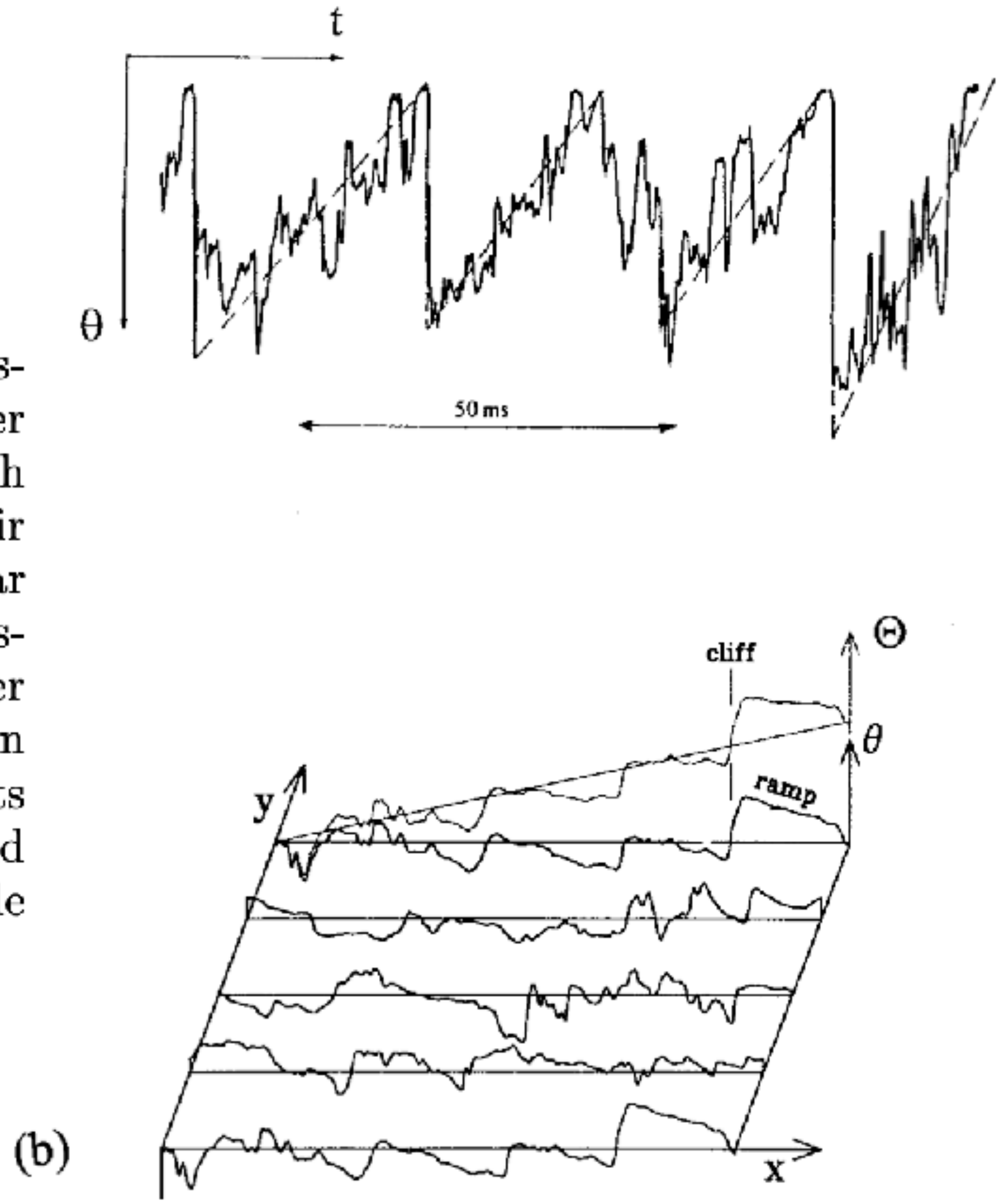
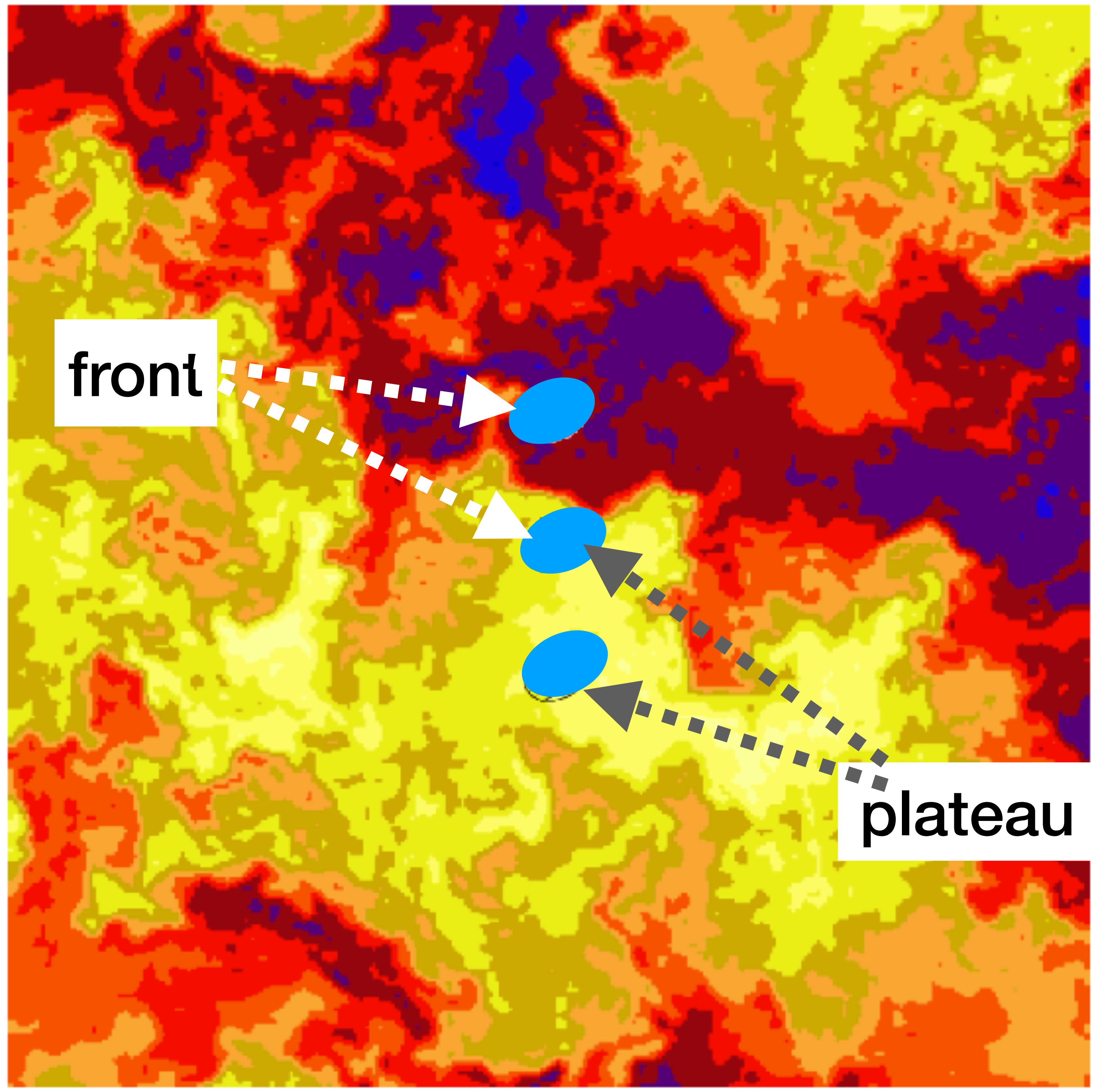
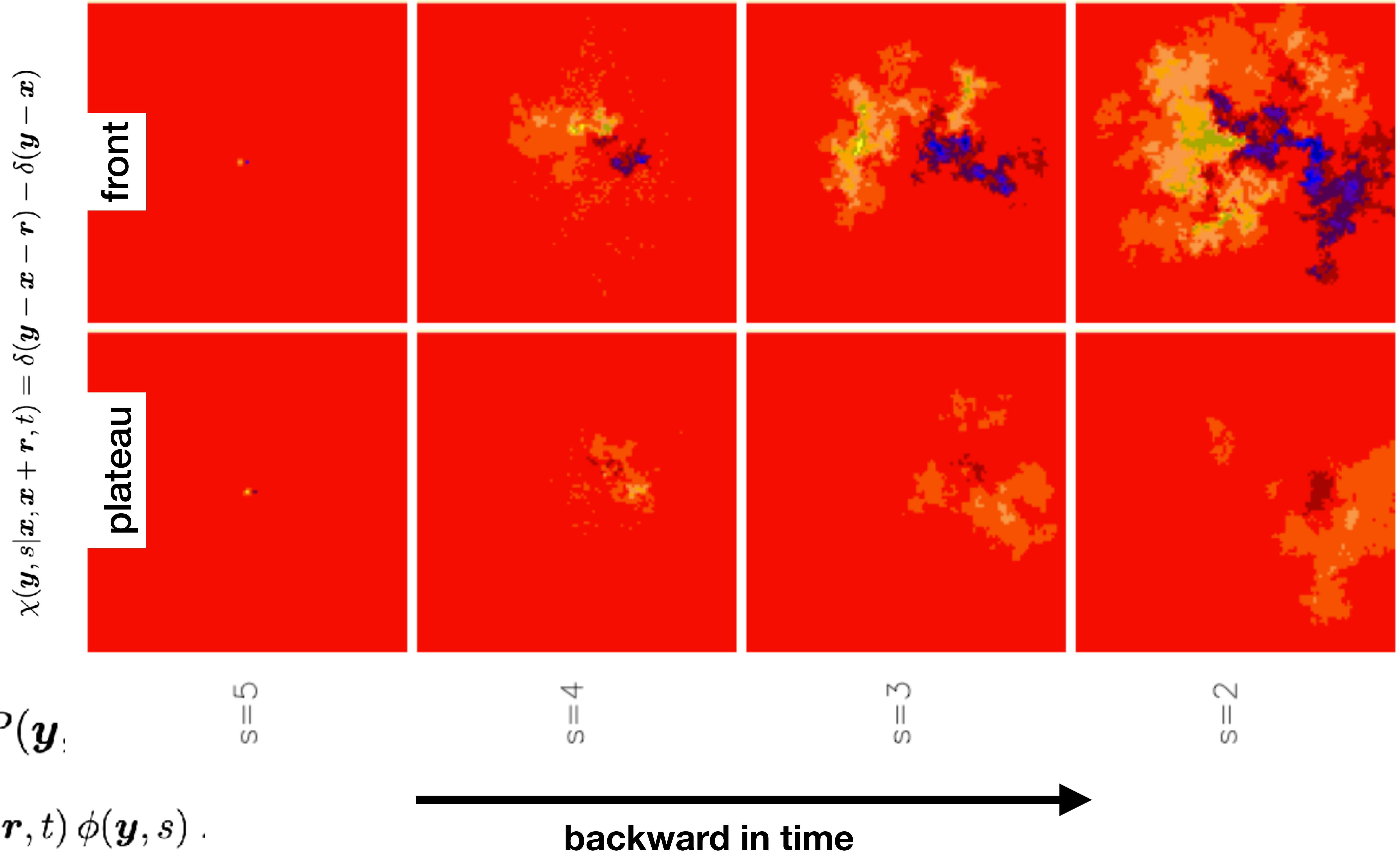


Figure 5 (a) A time series of temperature in a heated jet showing the ramp-cliff structure, from Sreenivasan et. al (1979). (b) Temperature spatial variation from numerical simulations of Holzer & Siggia (1994) with a mean temperature gradient. The full scalar is the top trace. The other traces are of the fluctuating component only.

Warhaft, Ann. Rev. Fluid Mech. **32**, 203 (2000)

Saturation of intermittency: Lagrangian origin





$$\chi(\boldsymbol{y}, s|\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{r}, t) = P(\boldsymbol{y}, s|\boldsymbol{x}+\boldsymbol{r}, t) - P(\boldsymbol{y},$$

$$\theta(\boldsymbol{x}+\boldsymbol{r}, t) - \theta(\boldsymbol{x}, t) = \int_0 \mathrm{d}s \int \mathrm{d}\boldsymbol{y}\; \chi(\boldsymbol{y}, s|\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{r}, t)\, \phi(\boldsymbol{y}, s)\,.$$

$$\chi(\boldsymbol{y}, s|\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{r}, t) = \delta(\boldsymbol{y}-\boldsymbol{x}-\boldsymbol{r}) - \delta(\boldsymbol{y}-\boldsymbol{x})$$

A. Celani, MC, A. Noullez Physica D 195, 283 (2004)

Remarks

At least in Kraichnan flows (and some evidence in realistic flows) we can say that the mechanism for anomalous scaling in passive scalar turbulence is the dominance of zero modes

Zero modes admit an interesting (and testable) physical interpretation: they are functions statistically preserved over the Lagrangian paths. This conservation comes from a compensation of the growth in scale by the geometry (shape)

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Zero modes admit an interesting (and testable) physical interpretation: they are functions statistically preserved over the Lagrangian paths. This conservation comes from a compensation of the growth in scale by the geometry (shape)

> limit of vanishing diffusion. The anomalies associated with statistically conserved quantities are qualitatively different from those produced by dynamically conserved quantities. For example, dissipation is a singular perturbation that breaks the conservation of dynamical integrals of motion and imposes a flux-constancy condition that is similar to quantum anomalies.[4] The flux constancy, in turn, is related to cascades of conserved quantities in the inertial range. Zero modes, in contrast, have no associated cascades, nor is their conservation broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. As different as they are, though, the two types of anomalies are intimately related: Flux constancy imposes certain scaling properties on the velocity field that generally lead to super-diffusion and to anomalous scaling of zero modes.

Falkovich & Sreenivasan. "Lessons from hydrodynamic turbulence." *Physics Today* 59.4 (2006): 43-49.

Remarks

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Zero modes admit an interesting (and testable) physical interpretation: they are functions statistically preserved over the Lagrangian paths. This conservation comes from a compensation of the growth in scale by the geometry (shape)

Anomalous scaling, for the velocity field (but it holds also for passive fields), is typically rationalized in terms of the multifractal model

The connection (if any) between these two views is an open question

Hidden symmetry (Talk by Chiara Calscibetta) would suggest anomalous scaling to originate from a multiplicative process ->multifractal

What is the connection with zero modes

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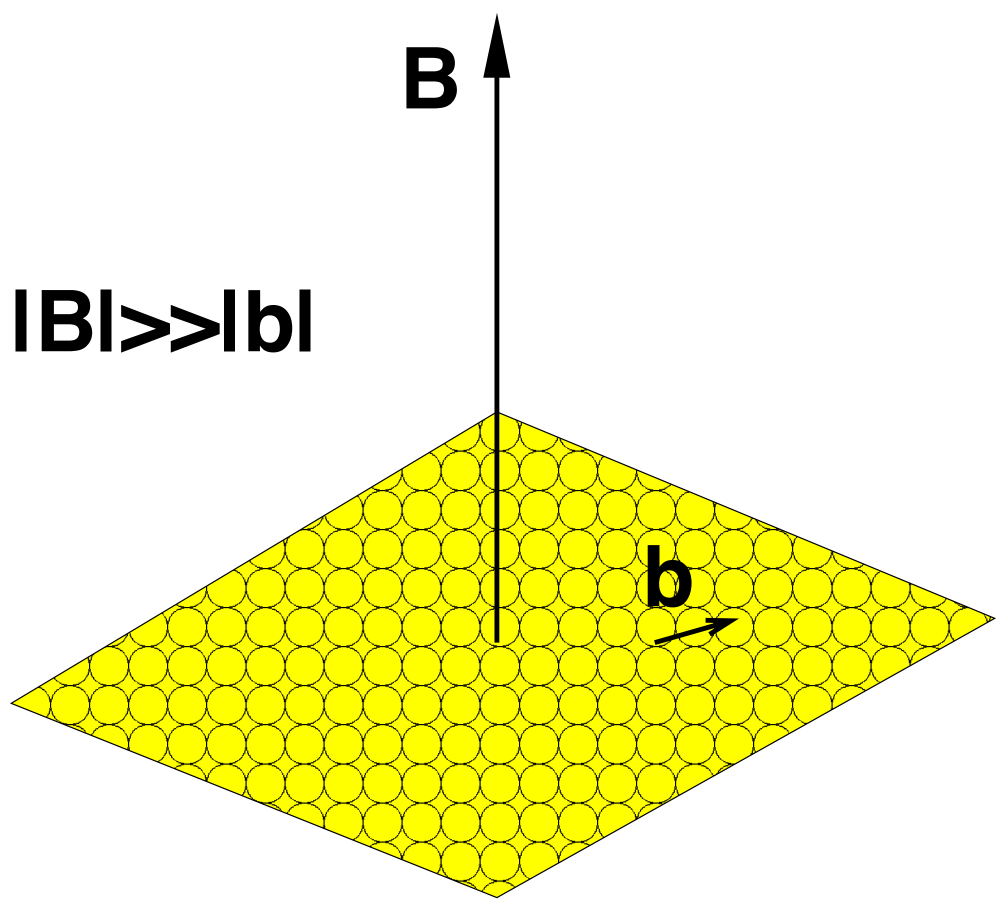
An example of active 2D MHD

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$ *magnetic potential*

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v} - \boxed{\Delta a\boldsymbol{\nabla} a}$$

$$\partial_t c + \boldsymbol{v}\cdot\boldsymbol{\nabla} c = \kappa\Delta c + F_c$$ *passive field*



$\boldsymbol{b} = \boldsymbol{\nabla}^{\perp} a = (-\partial_y a, \partial_x a)$ **Magnetic Field**

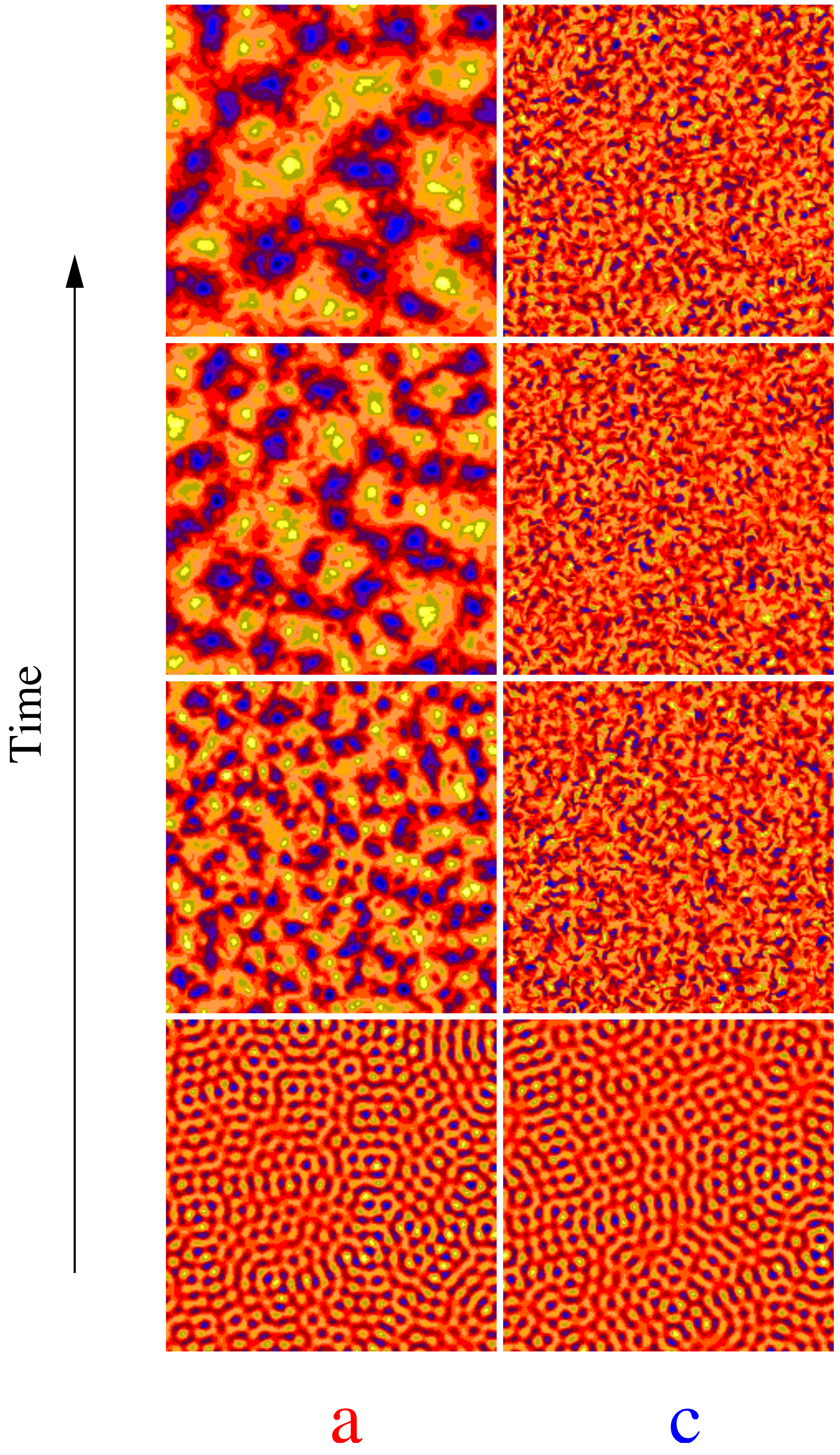
$(\boldsymbol{\nabla}\times\boldsymbol{b})\times\boldsymbol{b} = -\Delta a\boldsymbol{\nabla} a$ **Lorentz Force**

2d-mhd is obtained from 3d-mhd when the vertical magnetic field is much more intense that that on the transversal plane

We choose to force passive and active scalars in the same way

F_a & F_c are different realizations of the same random process that is δ-correlated in time and with support at scale $L_f \sim 1/k_f$

2D MHD: phenomenology



$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$ *magnetic potential*

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v} - \boxed{\Delta a \boldsymbol{\nabla} a}$$

$$\partial_t c + \boldsymbol{v}\cdot\boldsymbol{\nabla} c = \kappa\Delta c + F_c$$ *passive field*

Inverse cascade: no dissipative anomaly

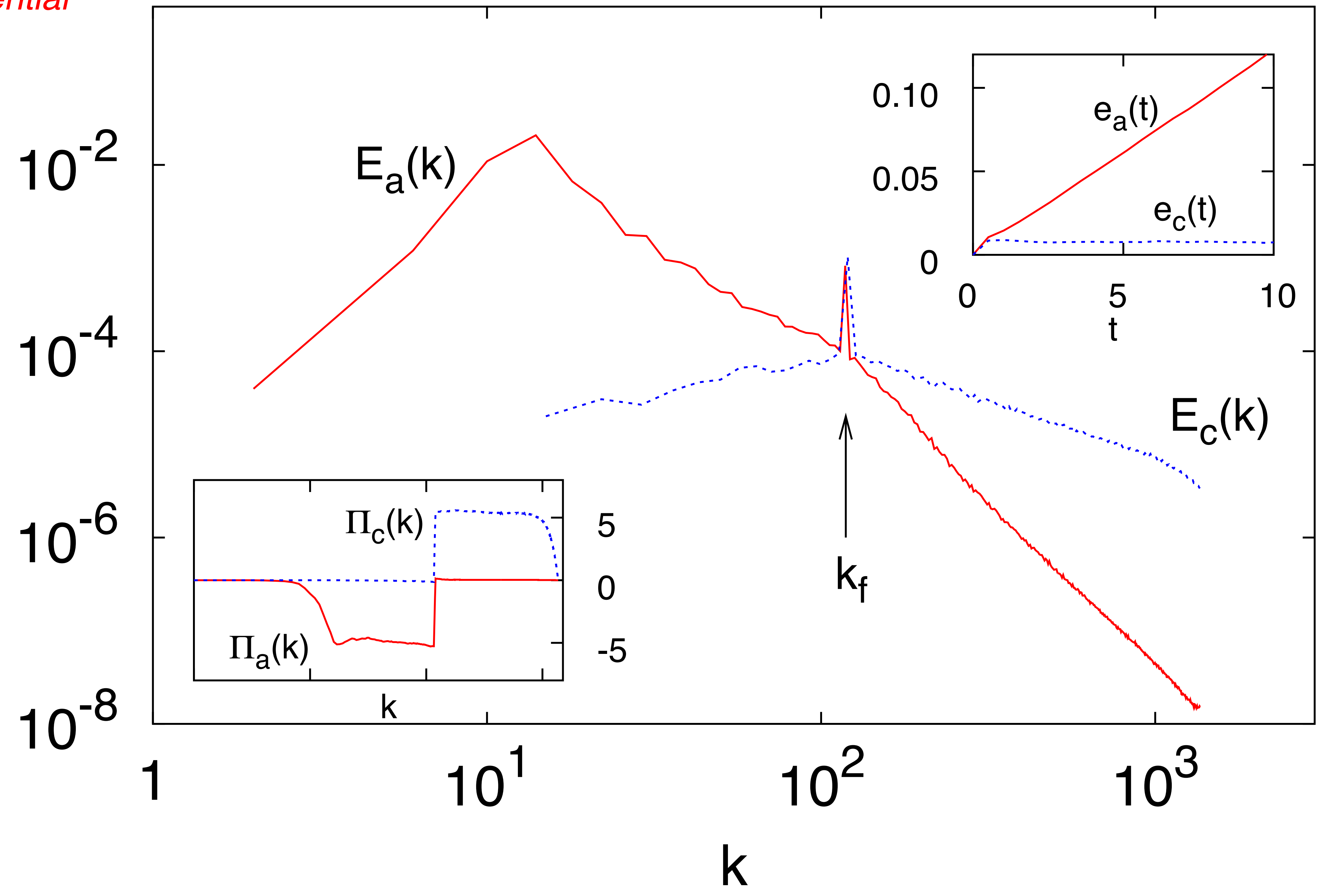
$$\epsilon_a = \lim_{\kappa\to 0} \kappa\langle|\boldsymbol{\nabla} a|^2\rangle = 0$$

$$e_a(t) = \frac{1}{2}\int a^2(\mathbf{x},t)\, d\mathbf{x} \propto t$$

Direct cascade: dissipative anomaly

$$\epsilon_c = \lim_{\kappa\to 0} \kappa\langle|\boldsymbol{\nabla} c|^2\rangle \approx \text{input}$$

$$e_c(t) = \frac{1}{2}\int c^2(\mathbf{x},t)\, d\mathbf{x} \approx const$$



As typical in inverse cascades the statistics of a (but also of v) is Gaussian and not intermittent

D. Biskamp and U. Bremer, Phys. Rev. Lett. **72**, 3819 (1994)

What is the origin of such differences?

Can we understand it adopting a Lagrangian point of view?

A. Celani, M.C., A. Mazzino & M. Vergassola, PRL **89**, 234502 (2002); NJP **6,** 72 (2004)

2D MHD: Another look at the absence of dissipative anomaly

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a$$

$$a(\boldsymbol{x}, t) = \left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s), s) \right\rangle_X = \int_0^t ds \int F_a(\boldsymbol{y}, s)\, p(\boldsymbol{y}, s|\boldsymbol{x}, t)\, d\boldsymbol{y}$$

$$a^2(\boldsymbol{x}, t) = \int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y}, s) F_a(\boldsymbol{y}', s') p(\boldsymbol{y}, s|\boldsymbol{x}, t) p(\boldsymbol{y}', s'|\boldsymbol{x}, t)$$

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Now let's see if we can find another expression for a^2

$$\partial_t a^2 + \boldsymbol{v}\cdot\nabla a^2 = \kappa\Delta a^2 + 2aF_a - \boxed{2\epsilon_a}$$

$$\epsilon_a \to 0$$

$$\Delta a^2 = 2a\Delta a - 2\boldsymbol{\nabla} a\cdot\boldsymbol{\nabla} a$$

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2D MHD: Another look at the absence of dissipative anomaly

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$$2\int_0^t ds \int_0^s ds' \Longrightarrow \int_0^t\int_0^t ds ds'$$

$$p(\boldsymbol{y}',s'|\boldsymbol{y},s)p(\boldsymbol{y},s|\boldsymbol{x},t) = p(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)$$

$$a^2(\boldsymbol{x},t) = \iint_0^t ds ds' \iint d\boldsymbol{y}d\boldsymbol{y}' p(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)F_a(\boldsymbol{y},s)F_a(\boldsymbol{y}',s')$$

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en passant *Integrating over x and averaging over forcing we can see that* $\langle a^2 \rangle \propto t$

2D MHD: Another look at the absence of dissipative anomaly

$$\int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y}, s) F_a(\boldsymbol{y}', s') p(\boldsymbol{y}, s|\boldsymbol{x}, t) p(\boldsymbol{y}', s'|\boldsymbol{x}, t)$$

$$\|$$

$$\int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y}, s) F_a(\boldsymbol{y}', s') p(\boldsymbol{y}, s; y, s'|\boldsymbol{x}, t)$$

2D MHD: Another look at the absence of dissipative anomaly

$$\left\langle \int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right\rangle^2 = \int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y},s)F_a(\boldsymbol{y}',s')p(\boldsymbol{y},s|\boldsymbol{x},t)p(\boldsymbol{y}',s'|\boldsymbol{x},t)$$

$$||$$

$$\int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y},s)F_a(\boldsymbol{y}',s')p(\boldsymbol{y},s;y,s'|\boldsymbol{x},t)$$

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$$\Vert$$

$$\left\langle \left(\int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right)^2 \right\rangle = \int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y},s) F_a(\boldsymbol{y}',s') p(\boldsymbol{y},s;y,s'|\boldsymbol{x},t)$$

2D MHD: Another look at the absence of dissipative anomaly

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$$\|$$

$$\left\langle \left(\int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t)\right)^2\right\rangle = \int_0^t ds \int_0^t ds' \int F_a(\boldsymbol{y},s)F_a(\boldsymbol{y}',s')p(\boldsymbol{y},s;y,s'|\boldsymbol{x},t)$$

$\epsilon_a \to 0$

One can generalize to

$$\left\langle \int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t)\right\rangle^N = \left\langle \left(\int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t)\right)^N\right\rangle$$

All paths are constrained to sum up the same forcing contribution!!!!!

2D MHD: A consequence of the absence of dissipative anomaly

2D MHD: A consequence of the absence of dissipative anomaly

$\epsilon_a \to 0$

$$\left\langle \int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right\rangle^N = \left\langle \left(\int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right)^N \right\rangle$$

How can all paths sum up the same forcing contribution?

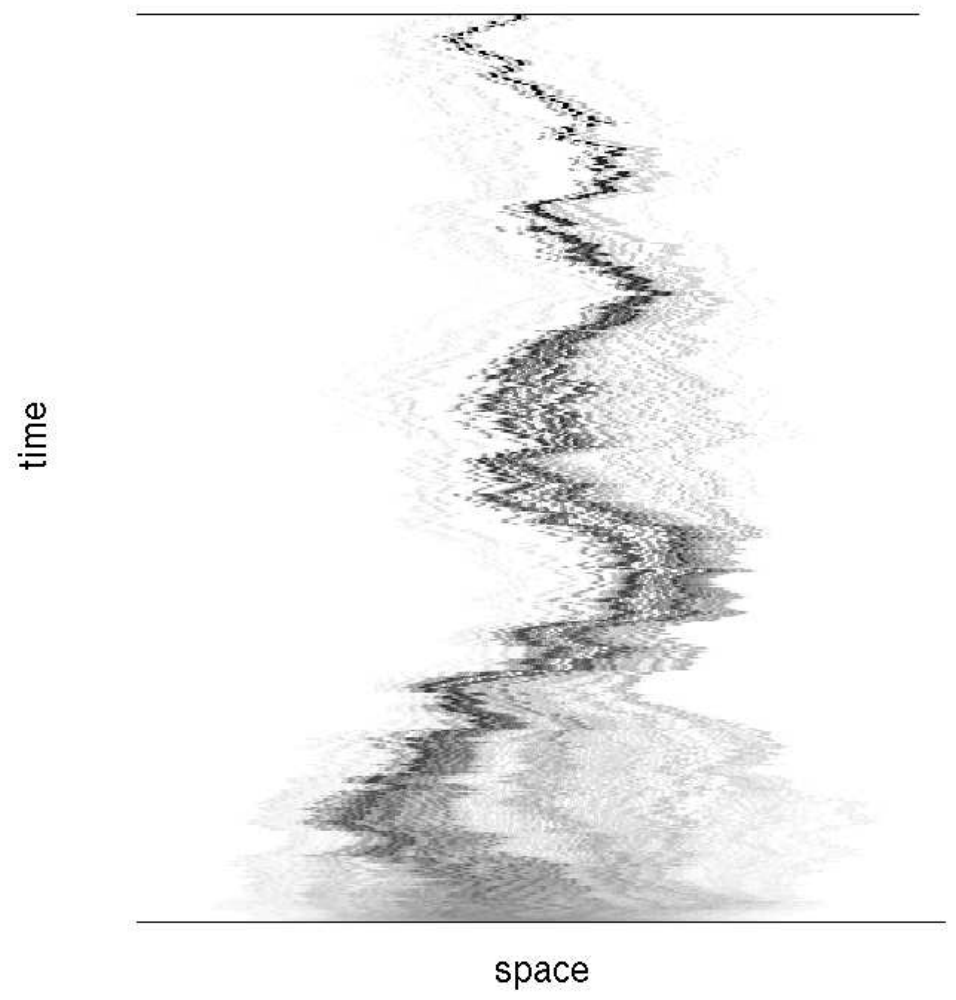
2D MHD: A consequence of the absence of dissipative anomaly

$\epsilon_a \to 0$

$$\left\langle \int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right\rangle^N = \left\langle \left(\int_0^t ds F_a(\boldsymbol{X}(s;\boldsymbol{x},t) \right)^N \right\rangle$$

How can all paths sum up the same forcing contribution?

In passive scalars
this happens in compressible flows
where for $\kappa \to 0$
all paths collapse



Gawedzki Vergassola, PhysicaD **138**, 63 (2000)

In our case velocity is incompressible
and the scalar field is active so the above relation
must be the result of the collective organization of many paths
and non-trivial correlations between the forcing and the paths

2D MHD: Lagrangian view

$$a(\mathbf{x},t) = \int_0^t ds \overbrace{\int p(\mathbf{y},s|\mathbf{x},t)\, F_a(\mathbf{y},s)\, d\mathbf{y}}^{\phi_a(s)}$$

$$c(\mathbf{x},t) = \int_0^t ds \overbrace{\int p(\mathbf{y},s|\mathbf{x},t)\, F_c(\mathbf{y},s)\, d\mathbf{y}}^{\phi_c(s)}$$

2D MHD: Lagrangian view

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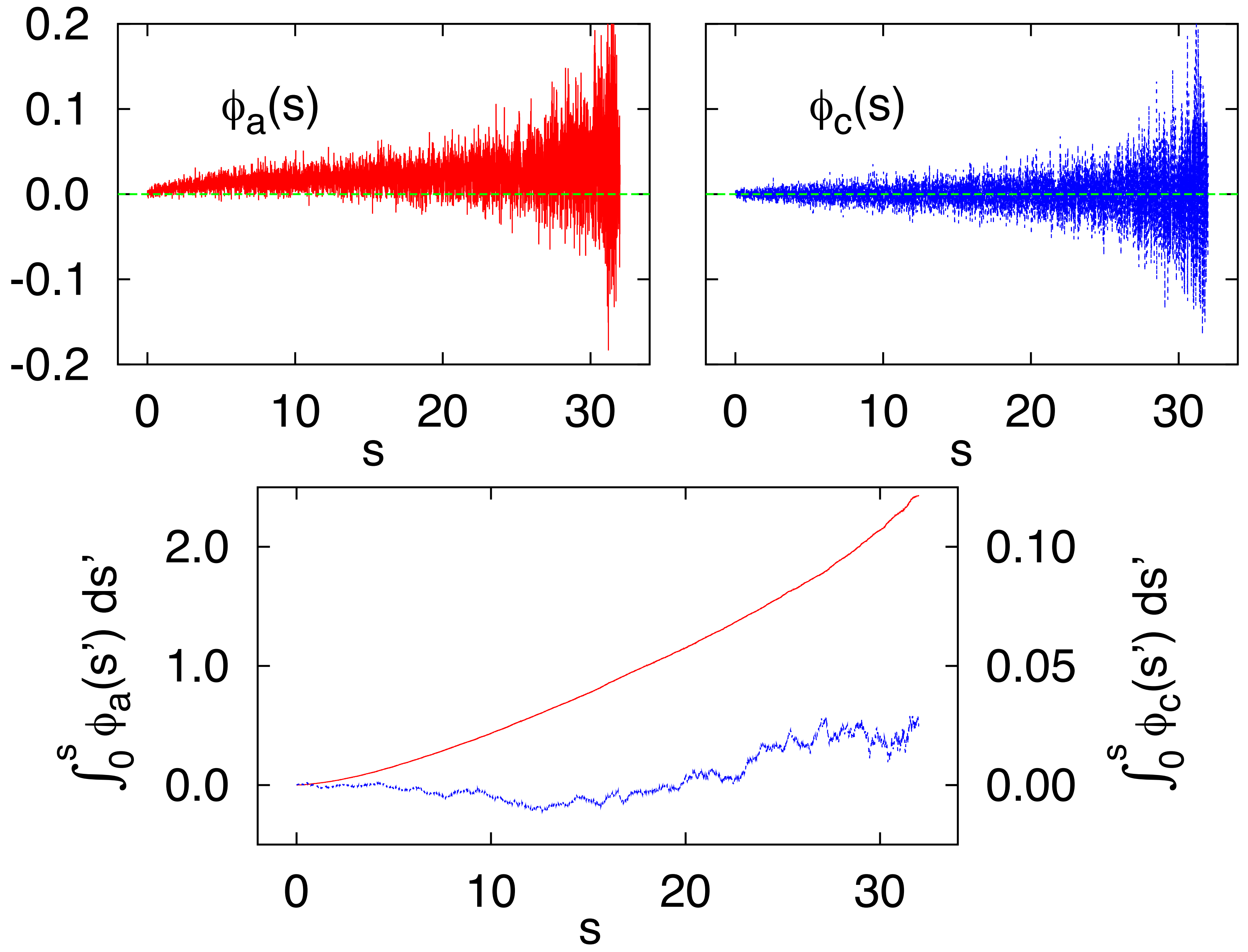
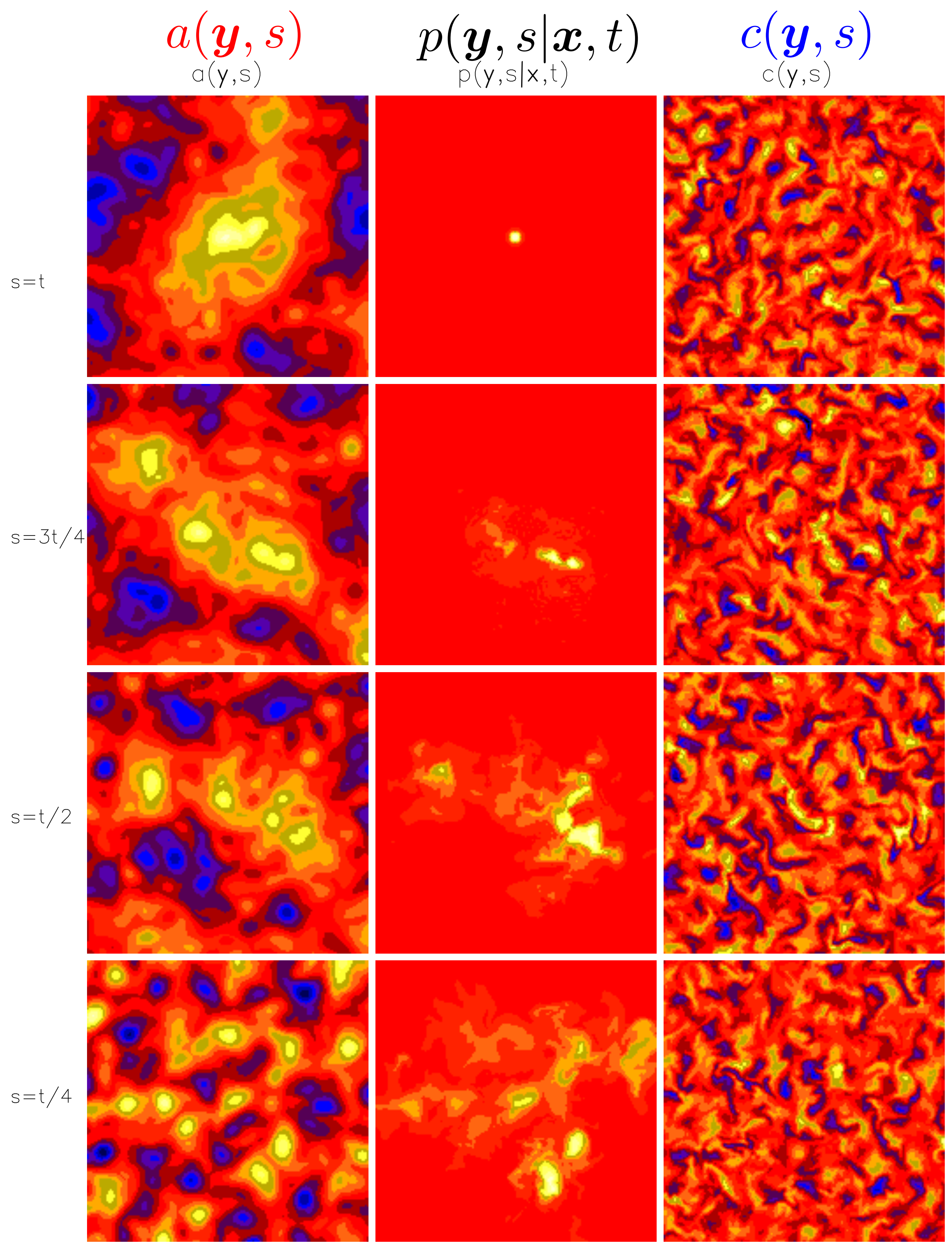
Correlations between F_a & Lagrangian paths





Some general considerations on active scalars

Fields that act on the velocity through local forces

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$
$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v} + \boxed{\mathcal{F}(a, \boldsymbol{\nabla} a, \ldots)}$$

e.g. Thermal Convection: a (temperature) $\mathcal{F} = -\beta g a$ (buoyancy) or MHD

Fields functionally linked to the velocity field

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$
$$v_i(\boldsymbol{x}, t) = \int d\boldsymbol{y}\, \Gamma_i[\boldsymbol{x}, \boldsymbol{y}]\, a(\boldsymbol{y}, t)$$

e.g. 2d-NS $a = \boldsymbol{\nabla}\times\boldsymbol{v}$ (vorticity) $\Gamma_i[\boldsymbol{x}, \boldsymbol{y}] = -(2\pi)^{-1}\,\epsilon_{ij}\partial_j \log|\boldsymbol{x}-\boldsymbol{y}|$

or Surface Quasi-Geostrophic equation $v_i(\boldsymbol{x}, t) = \int d\boldsymbol{y}\, a(\boldsymbol{y}, t)\, \epsilon_{ij}\partial_{x_i}|\boldsymbol{x}-\boldsymbol{y}|^{-1}$

Due to the “activity” i.e. the dependence of v on a the problem is non linear and one cannot invoke the zero modes picture to justify universality of the statistics in other terms both the statistical properties of velocity and scalar fields are no more guarantee to be universal with respect to the forcing

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