Lagrangian Turbulence: from tracers to intermittency and transport (II)

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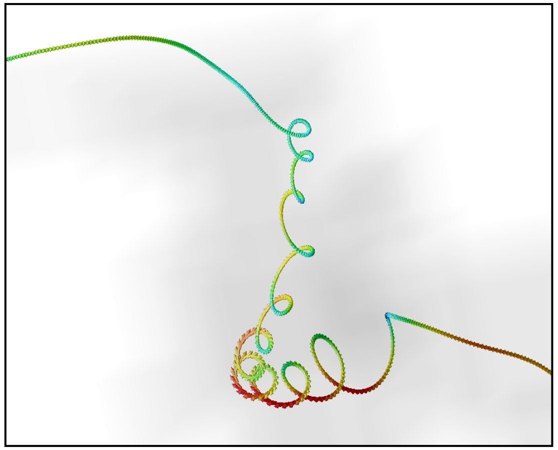
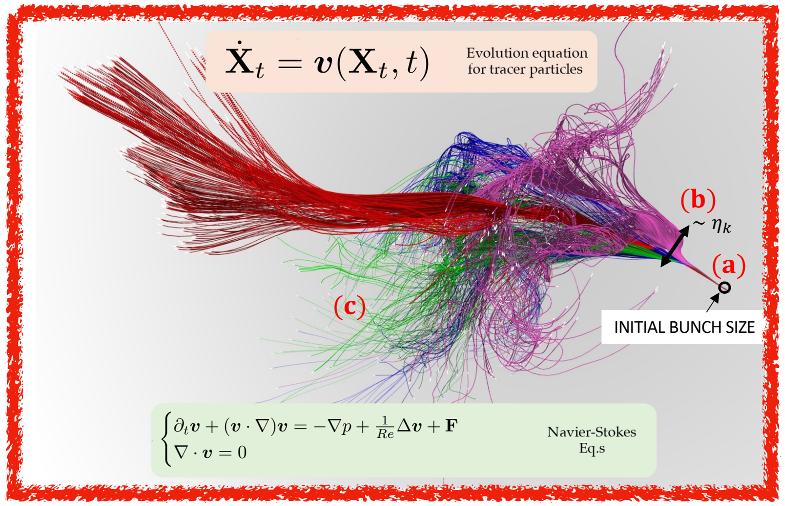
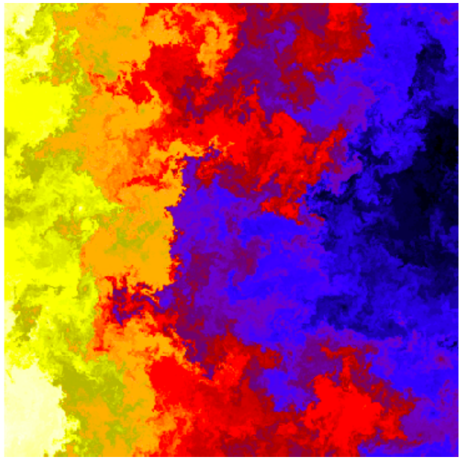






Outline -topic 2-

- Brief recall of single particle dispersion in turbulence
- Relative dispersion of two particles: Chaos, Richardson dispersion + intermittency, explosive separation
- Some ideas on multiparticle dispersion, shape and size evolution, zero modes

Single particle: absolute dispersion

$$\dot{\boldsymbol{X}} = \boldsymbol{u}(\boldsymbol{X}, t) + \sqrt{2D_0}\boldsymbol{\eta}(t)$$

We consider small particles with same density as the fluid: no inertia

Gaussian noise $\langle \eta_i(t) \rangle = 0 \quad \langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t-t')$

A simple limiting case

no flow (Brownian diffusion)

$$\dot{\boldsymbol{X}} = \sqrt{2D_0}\boldsymbol{\eta}(t)$$

$$X_i(t) - X_i(0) = \sqrt{2D_0}\int_0^t ds\, \eta_i(s)$$

$$\langle [X_i(t) - X_i(0)]^2 \rangle = 2D_0 \int_0^t ds \int_0^t ds' \underbrace{\langle \eta_i(s)\eta_i(s') \rangle}_{=\,\delta(s-s')} = 2D_0 t$$

Single particle: absolute dispersion

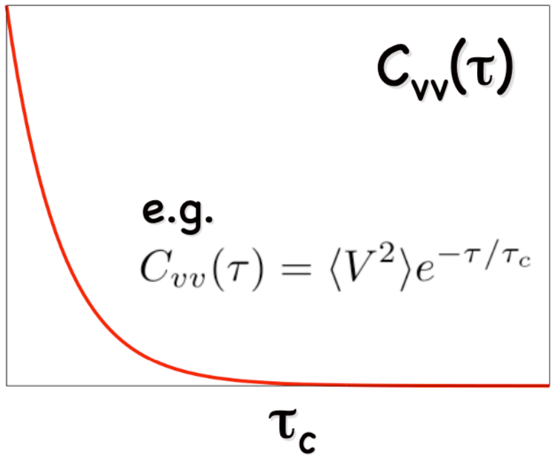
Another limit: no noise (only advection)

$$\dot{\boldsymbol{X}} = \boldsymbol{V}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t) \qquad X_i(t) - X_i(0) = \int_0^t ds\, V_i(s) \qquad \langle [X_i(t) - X_i(0)]^2 \rangle = \int_0^t ds \int_0^t ds' \, \langle V_i(s) V_i(s') \rangle$$

Lagrangian Correlation function

$$\langle V_i(s) V_i(s') \rangle = C_{ii}(s, s') \overset{\text{by stationarity}}{=} C_{ii}(s - s')$$

$$\langle [X_i(t) - X_i(0)]^2 \rangle = \int_0^t ds \int_0^t ds' \, C_{ii}(s - s') = 2 \int_0^t ds \int_0^s ds' C_{ii}(s - s') = 2 \int_0^t ds \int_0^s d\tau C_{ii}(\tau)$$



Lagrangian correlation time

$$\tau_c = \frac{\int_0^\infty d\tau C_{ii}(\tau)}{C_{ii}(0)}$$

$$\langle [X_i(t) - X_i(0)]^2 \rangle = \begin{cases} C(0) t^2 & t \ll \tau_c \\ 2[C(0)\tau_c] t & t \gg \tau_c \end{cases}$$

$$C(0) = \langle V^2 \rangle \qquad D^E = C(0)\tau_c$$

Eddy diffusivity

Single particle: superdiffusion

What does happen if the correlation function is not integrable? i.e. if the correlation time is not finite

$$\tau_c = \frac{\int_0^\infty d\tau C_{ii}(\tau)}{C_{ii}(0)} = \infty$$

$$\Downarrow$$

Superdiffusion (Anomalous diffusion)

$$\langle [X_i(t) - X_i(0)]^2 \rangle \sim t^\alpha \quad \alpha > 1$$

In turbulence the correlation time is finite and thus
we do not expect anomalous diffusion for the single particle
But the long time diffusive behavior will be much faster than in the absence of flow
due to eddy diffusivity $D^E \sim \tau_c U^2$

Relative dispersion

$$\dot{\boldsymbol{X}}_1 = \boldsymbol{u}(\boldsymbol{X}_1, t)$$
$$\dot{\boldsymbol{X}}_2 = \boldsymbol{u}(\boldsymbol{X}_2, t)$$

$$\boldsymbol{R} = \boldsymbol{X}_2 - \boldsymbol{X}_1$$

$$\dot{\boldsymbol{R}} = \boldsymbol{u}(\boldsymbol{X}_1 + \boldsymbol{R}, t) - \boldsymbol{u}(\boldsymbol{X}_1, t) = \delta_{\boldsymbol{R}} \boldsymbol{u}$$

Relative dispersion depends on the two-point properties of the velocity field

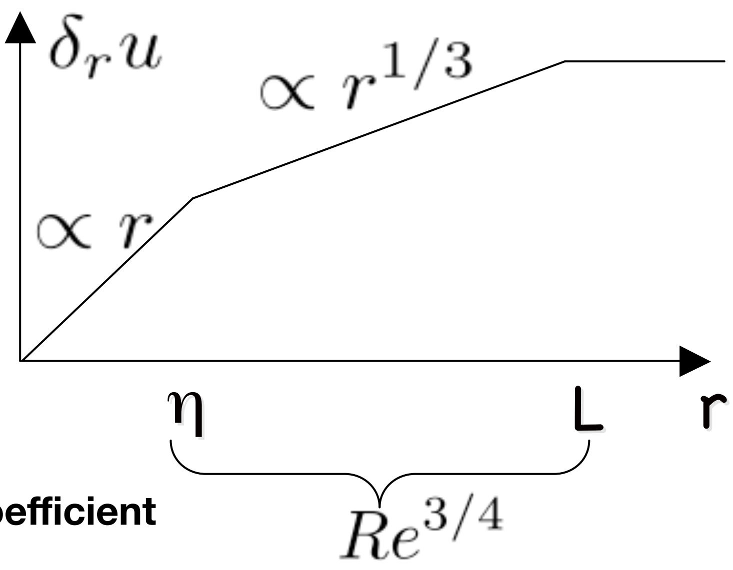
Different behaviors depending on the initial separation

$$R = |\boldsymbol{R}|$$

I $\quad R \ll \eta \rightarrow \delta_R u \propto R$

II $\quad \eta \ll R \ll L \rightarrow \delta_R u \propto R^{1/3}$

III $\quad R \gg L \rightarrow \delta_L u \quad (Gaussian\ uncorrelated\ variables)$



The third regime is like Brownian diffusion with an enhanced diffusion coefficient
what about the other two regimes?

Relative dispersion: small separation

$R = |\boldsymbol{R}| \ll \eta \qquad \dot{\boldsymbol{R}} = \mathbb{A}\boldsymbol{R} \qquad \mathbb{A}_{ij} = \frac{\partial u_i}{\partial x_j} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \mapsto Tr[\mathbb{A}] = 0$

In 3D even for non turbulent flows (i.e. laminar flows) particles separate exponentially due to chaos!

$$R(t) \approx R(0)e^{\lambda_1 t} \qquad \lambda_1 > 0$$

the maximal Lyapunov Exponent

$$\boldsymbol{R}(t) = \mathbb{W}(0,t)\boldsymbol{R}(0) \qquad \mathbb{W}(0,t) = \mathcal{T}\exp\left[\int_0^t \mathbb{A}(s)ds\right]$$

$$\Big[\underbrace{\mathbb{W}^\dagger(0,t)\mathbb{W}(0,t)}_{\text{positive and symmetric}}\Big]^{1/2} = \mathbb{V}(\boldsymbol{x}_0,t) = \mathbb{Q}(\boldsymbol{x}_0,t)\mathbb{D}(\boldsymbol{x}_0,t)\mathbb{Q}^\dagger(\boldsymbol{x}_0,t) \qquad \mathbb{D}(\boldsymbol{x}_0,t) = \mathrm{diag}\{e^{t\gamma_1(\boldsymbol{x}_0,t)},\ldots,e^{t\gamma_d(\boldsymbol{x}_0,t)}\}$$

finite time Lyapunov exponents

Oseledec Theorem

$$\gamma_i(\boldsymbol{x}_0,t) \xrightarrow[t\to\infty]{} \lambda_i(\boldsymbol{x}_0) = \lambda_i$$

if ergodic

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$$

Lyapunov exponents

Relative dispersion: small separation

λ_1 => growth rate of infinitesimal segments

$\lambda_1+\lambda_2$ => growth rate of infinitesimal surfaces

$\lambda_1+\lambda_2+\lambda_3$ => growth rate of infinitesimal volumes

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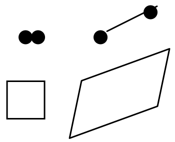


$$L(t) = L(0)e^{\lambda_1 t}$$

$$A(t) = A(0)e^{(\lambda_1+\lambda_2)t}$$

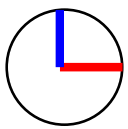
incompressibility

$$\dot{\boldsymbol{R}} = \mathbb{A}\boldsymbol{R} \qquad \mathbb{A}_{ij} = \frac{\partial u_i}{\partial x_j} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \mapsto Tr[\mathbb{A}] = 0$$

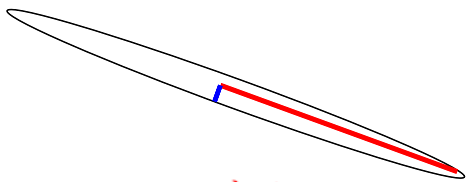
conservation of volumes

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

Chaotic stretching



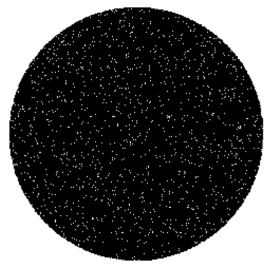
$$L_1(0) = L_2(0) = \epsilon$$



$$L_1(t) = \epsilon e^{\lambda_1 t}$$

$$L_2(t) = \epsilon e^{-|\lambda_2| t}$$

Nonlinear effects lead to folding



Stretching and folding are at the base of mixing in laminar flows

Relative dispersion: small separation

Finite Time Lyapunov Exponent

$$\gamma_1(t) = \frac{1}{t}\log\left(\frac{R(t)}{R(0)}\right)$$

Law of large numbers (Oseledec)

$$\lim_{t\to\infty}\lim_{R(0)\to 0}\gamma_1(t) = \lambda_1$$

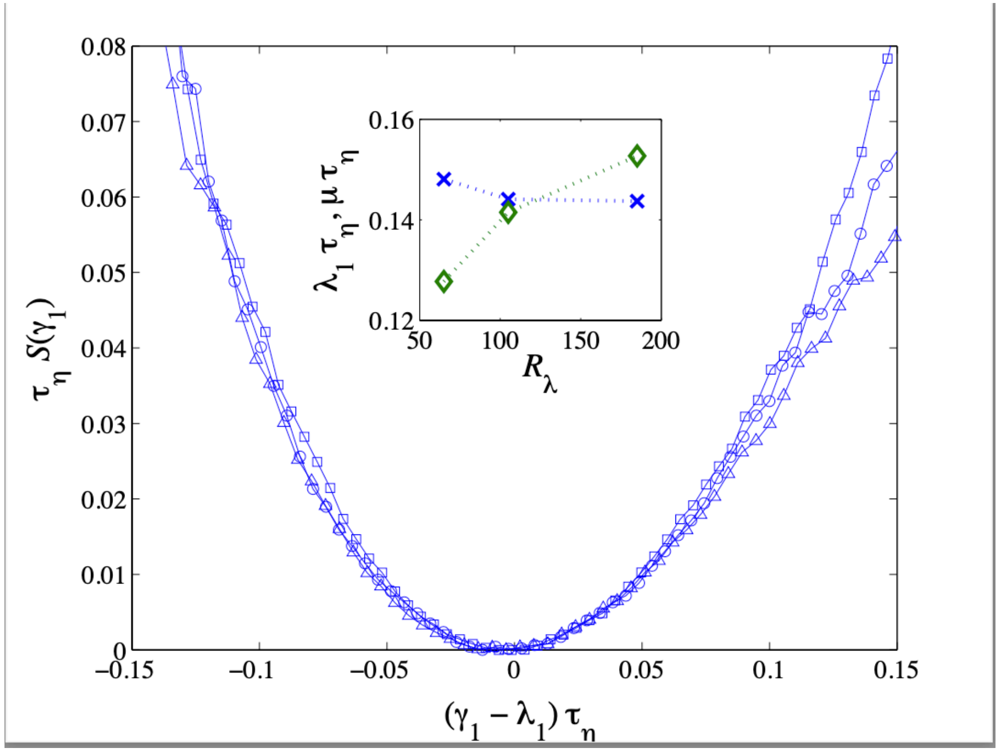
Large deviation theory

$$P_t(\gamma) \sim e^{-S(\gamma)t}$$

$$S(\gamma) > 0 \quad if \quad \gamma \neq \lambda_1 \quad \& \quad S(\lambda_1) = 0$$

For large times and close to the minimum (TLC)

$$P_t(\gamma) \sim \exp\left[-\frac{t(\gamma-\lambda_1)^2}{2\sigma^2}\right]$$



J. Bec et al. *Physics of Fluids* 18, 091702 (2006).

Relative dispersion: small separation

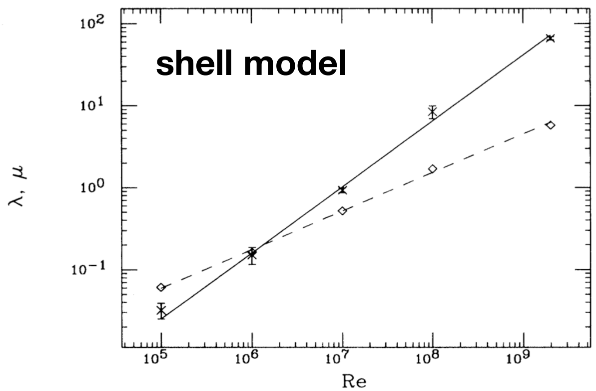
K41 D. Ruelle, Phys. Lett. 72A, 81 (1979)

$$\lambda_1 \sim \frac{1}{\tau_\eta} \sim \frac{1}{T_L}\left(\frac{T_L}{\tau_\eta}\right) \sim Re^{1/2} \sim Re_\lambda$$

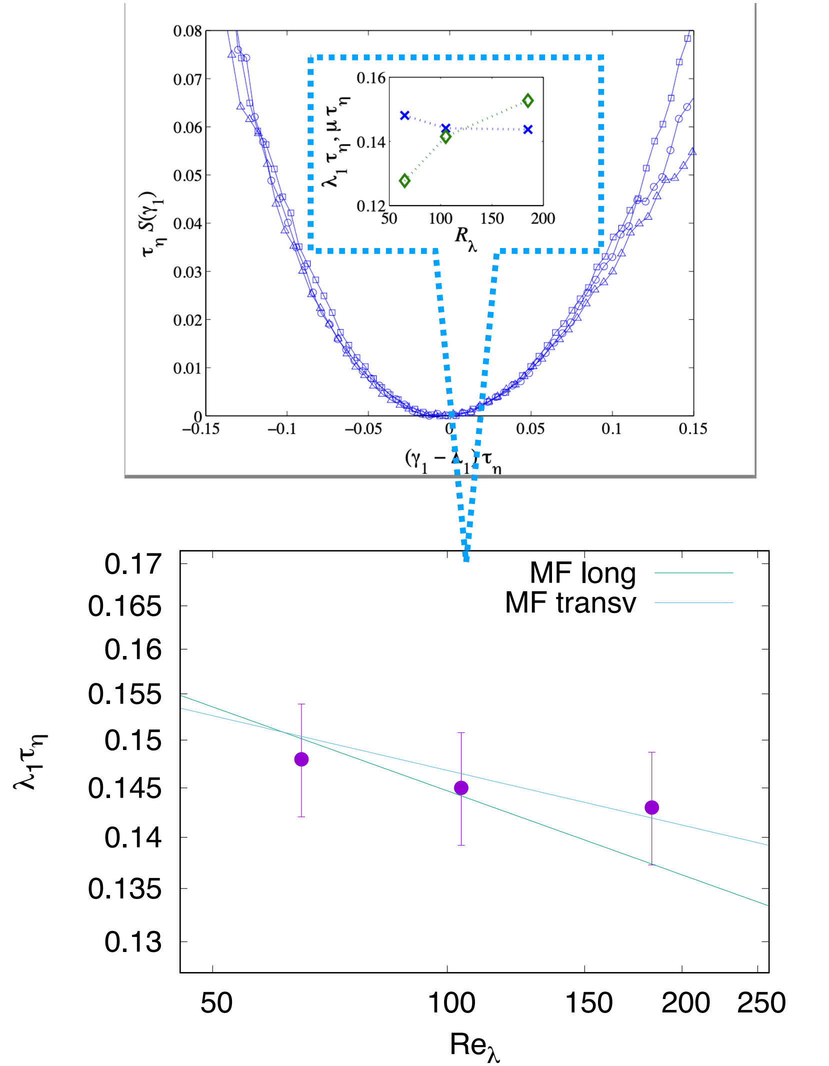
MF A. Crisanti, M.H. Jensen, A. Vulpiani, & G. Paladin, PRL *70*, 166 (1993)

$$\lambda_1 \sim \frac{1}{\tau_\eta} \sim \frac{1}{T_L}\left(\frac{T_L}{\tau_\eta}\right) \sim Re^{\frac{1-h}{1+h}} \qquad P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} \sim Re^{\frac{D(h)-3}{1+h}}$$

$$\lambda_1 \sim Re^{\chi} \qquad \chi = \max_h \left\{ \frac{D(h)-2-h}{1+h} \right\}$$



A. Crisanti, M.H. Jensen, A. Vulpiani, G. Paladin. *PRL* 70, 166 (1993)



J. Bec et al. *Physics of Fluids* 18, 091702 (2006).

Dispersion in the inertial range: Richardson

The history of relative dispersion in turbulence starts with Richardson (1926) predating K41

From experimental observations of the evolution of the separation of balloons in the atmosphere

$$D_{turbo}(R) \sim R^{4/3}$$

$$\partial_t P = R^{-2}\partial_R[D(R)R^2\partial_R P]$$

$$R' = \lambda R \qquad t' = \lambda^a t$$

$$\lambda^{-a}\partial_{t'} P = \lambda^{-2/3} R'^{-2}\partial_{R'}[D(R')R'^2\partial_{R'} P]$$

invariant if $a = 2/3 \Longrightarrow R \sim t^{3/2}$

so we expect

$$\langle R^2(t)\rangle \sim t^3$$

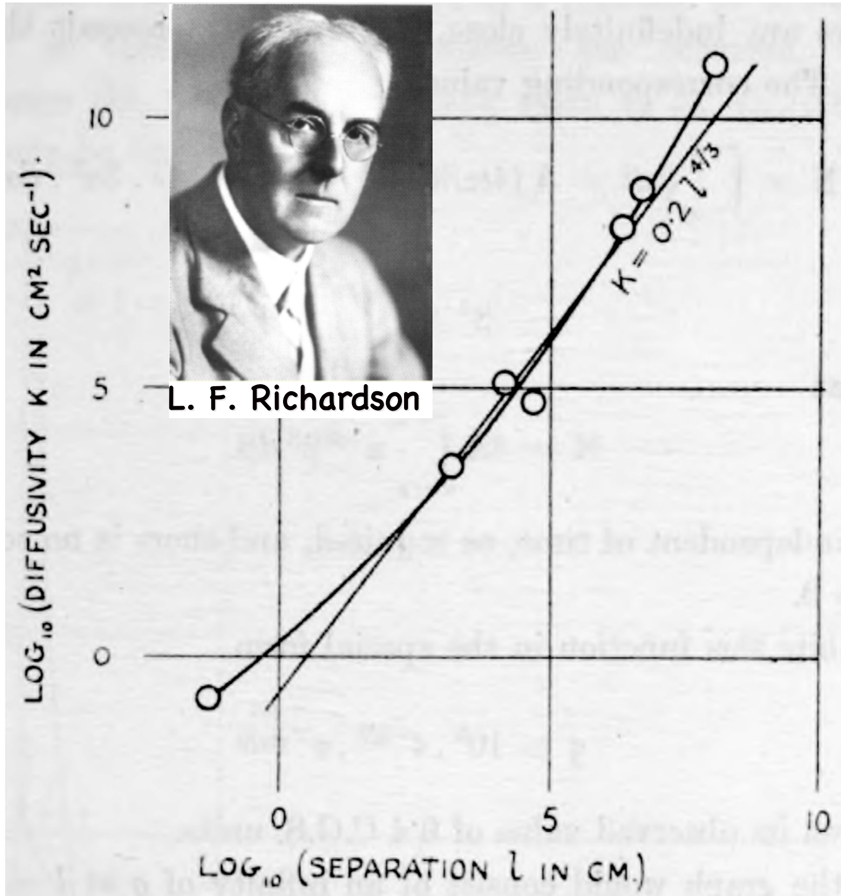


Fig. 5. Figure from the original Richardson's paper [8] representing the experimental results of the diffusivity D measured as a function of l, i.e. t
distance separation between two particles. Starting from these data, Richardson proposed his famous 4/3 law drawn in the figure.
Source: Reproduced from [8].

L.F. Richardson, Proc. Roy. Soc. A 110, 709 (1926).

Dispersion in the inertial range: Richardson

$$\dot{\boldsymbol{R}} = \boldsymbol{u}(\boldsymbol{X}_1 + \boldsymbol{R}, t) - \boldsymbol{u}(\boldsymbol{X}_1, t) = \delta_{\boldsymbol{R}} \boldsymbol{u}$$

$$\frac{\partial R^2}{\partial t} = 2\boldsymbol{R} \cdot \boldsymbol{\delta_R u} \sim 2R(\epsilon R)^{1/3} = 2\epsilon^{1/3} R^{4/3}$$

Consistent with K41

$$D_{turbo}(R) \sim R^{4/3}$$

$$\partial_t P = R^{-2} \partial_R [D(R) R^2 \partial_R P]$$

$$D(R) = \mathcal{D}\epsilon^{1/3} R^{4/3}$$

$$P(R,t) \propto \frac{1}{(\mathcal{D}\epsilon^{1/3} t)^{9/2}} \exp\left(-\frac{9R^{2/3}}{4\mathcal{D}\epsilon^{1/3} t}\right)$$

$R^2 R^2 dR$

$$\langle R^2(t) \rangle = g\epsilon t^3$$

NB: the underlying assumption of Richardson diffusion equation is time decorrelation

In analogy with single particle dispersion we can write $\langle (R(t) - R(0))^2 \rangle = \int_0^t ds \int_0^t ds' \langle \delta_R v(s) \delta_R v(s') \rangle$

$$\langle R^2(t) \rangle^{2/3} \sim t^2 \sim \frac{\partial \langle R^2 \rangle}{\partial t} = 2 \int_0^t \langle \delta_R v(t) \delta_R v(s) \rangle = 2\tau_c(t) \langle \delta_R v^2 \rangle$$

$$\langle \delta_R v^2 \rangle \sim \langle R^2(t) \rangle^{1/3} \sim t$$

$$\tau_c(t) \sim \langle R^2(t) \rangle^{1/3} \sim t$$

which implies that $\delta_R v$ remains correlated invalidating the approach

G. Falkovich, K. Gawe¸dzki, M. Vergassola Rev. Mod. Phys. 73, 913 (2001)

Dispersion in the inertial range: Richardson

Persistence of correlations

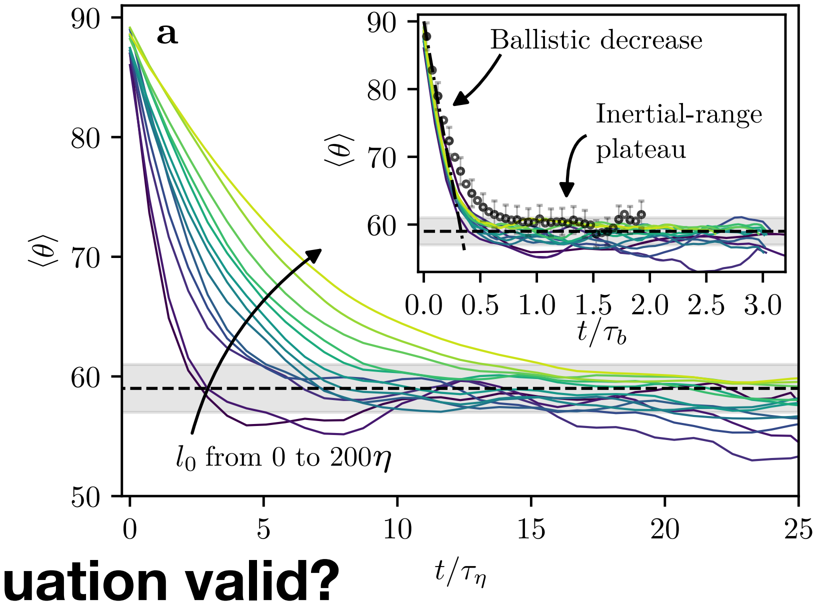
Shnapp, et al (2023). *Nature Commu*, *14*(1), 4195.

$$\dot{\boldsymbol{R}} = \boldsymbol{u}(\boldsymbol{X}_1 + \boldsymbol{R}, t) - \boldsymbol{u}(\boldsymbol{X}_1, t) = \delta_{\boldsymbol{R}}\boldsymbol{u}$$

$$\frac{\partial R^2}{\partial t} = 2\boldsymbol{R}\cdot\boldsymbol{\delta_R u}$$

$$\frac{\partial R}{\partial t} = \frac{\boldsymbol{R}\cdot\boldsymbol{\delta_R u}}{R}$$

$$\cos\theta = \frac{\boldsymbol{R}\cdot\boldsymbol{\delta_R u}}{R\delta_R u} = \frac{1}{\delta_R u}\frac{\partial R}{\partial t}$$



When is the diffusive equation valid?

I. M. Sokolov, PRE 60.5 (1999): 5528.

$$\langle(\delta_R v)^2\rangle \sim (\delta_L v)^2 \left(\frac{R}{L}\right)^\alpha$$

$$\tau_R \sim T_L \left(\frac{R}{L}\right)^\beta$$

$$\frac{\alpha}{2} + \beta < 1$$

$$\partial_t P = R^{-2}\partial_R[D(R)R^2\partial_R P]$$

$$D(R) \sim R^{\alpha+\beta}$$

Richardson

$\alpha = \beta = 2/3$ **is marginal!**

Diffusion eq holds if the displacement from R to R+Δ_R in a time τ_R is much smaller than R

$$\Delta_R \sim \delta_R u \tau_R \sim \delta_L u T_L \left(\frac{R}{L}\right)^{\alpha/2+\beta}$$

Richardson diffusion: refined view

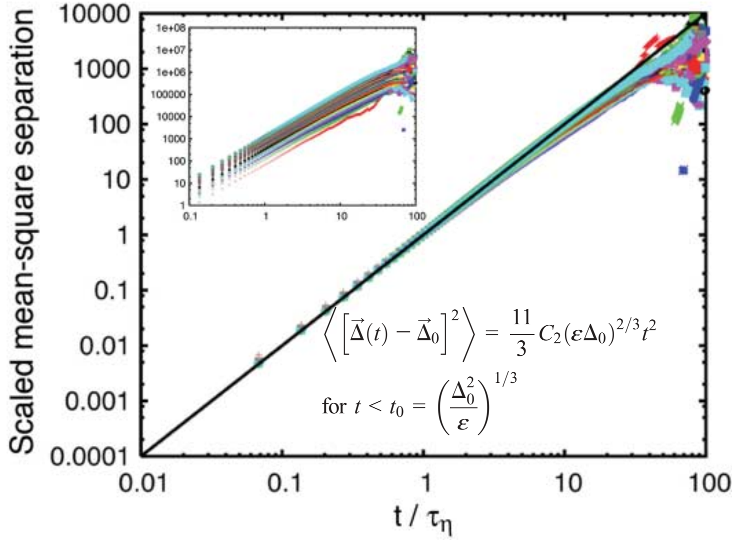
So far we ignored the initial separation $R_0 = R(0)$ **Typical time at scale R** $\tau(R) \sim \frac{R}{\delta_R u} \sim T_L \left(\frac{R}{L}\right)^{2/3}$

So we should expect that if we start with $R_0 \in [\eta, L]$ **the initial velocity** $\delta_{R_0} u$ **will be "maintained" for a time** $\tau(R_0)$

Therefore $0 < t < \tau(R_0)$ **we should expect a "ballistic" regime** $\langle R^2(t) \rangle \approx \langle (\delta_{R_0} u)^2 \rangle \, t^2$ **(so-called Batchelor dispersion)**

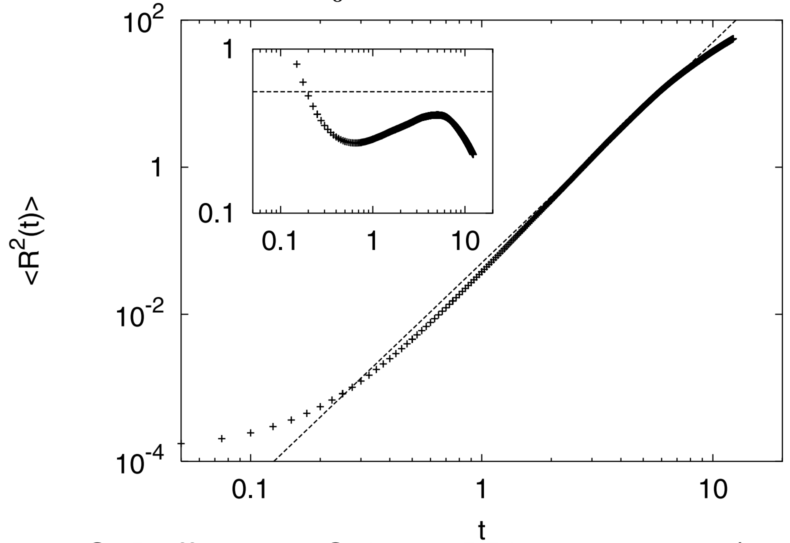
This adds some difficulties in observing Richardson dispersion for large initial separations

EXP R_0 in the inertial range



M Bourgoin, NT. Ouellette, H Xu, J. Berg, E. Bodenschatz Science 311, 835 (2006)

DNS $R_0 \approx \eta$



G. Boffetta & I. Sokolov PRL 88, 094501 (2002)

Richardson diffusion: refined view

N. Ouellette, et al NJP 8, 109 (2006)

$$\langle|\boldsymbol{R}(t)-\boldsymbol{R}_0|^2\rangle = \langle(\delta_{R_0}\boldsymbol{u}\cdot\delta_{R_0}\boldsymbol{u})\rangle t^2 + \langle\delta_{R_0}\boldsymbol{u}\cdot\delta_{R_0}\boldsymbol{a}\rangle t^3 + o(t^3)$$

$\approx -2\epsilon$

J. Mann, S. Ott, & JS Andersen (1999)

R.J. Hill JoT 7, N43 (2006)

$$t_0 = S_2(r_0)/(2\epsilon)$$

time scale for the end of the ballistic regime

The data collapse extends to times larger than t_0 when the mean-squared separation tends to Richardson t^3 regime. This unexpected fact implies that t_0 is not only the time scale of departure from the ballistic regime, but also that of convergence to Richardson's law. More precisely, numerical data suggest that for $t \gg t_0$

$$\langle|\delta\boldsymbol{x}(t)-\delta\boldsymbol{x}(0)|^2\rangle_{r_0} = g\epsilon t^3\,[1+Ct_0/t] + \text{h.o.t.} \qquad (4)$$

C does not strongly depend on the Reynolds number. Sys-

But depends on the initial separation!

$$C = C(R_0)$$

$$C = 0 \quad for \quad R_0 \approx 4\eta$$

Optimal choice

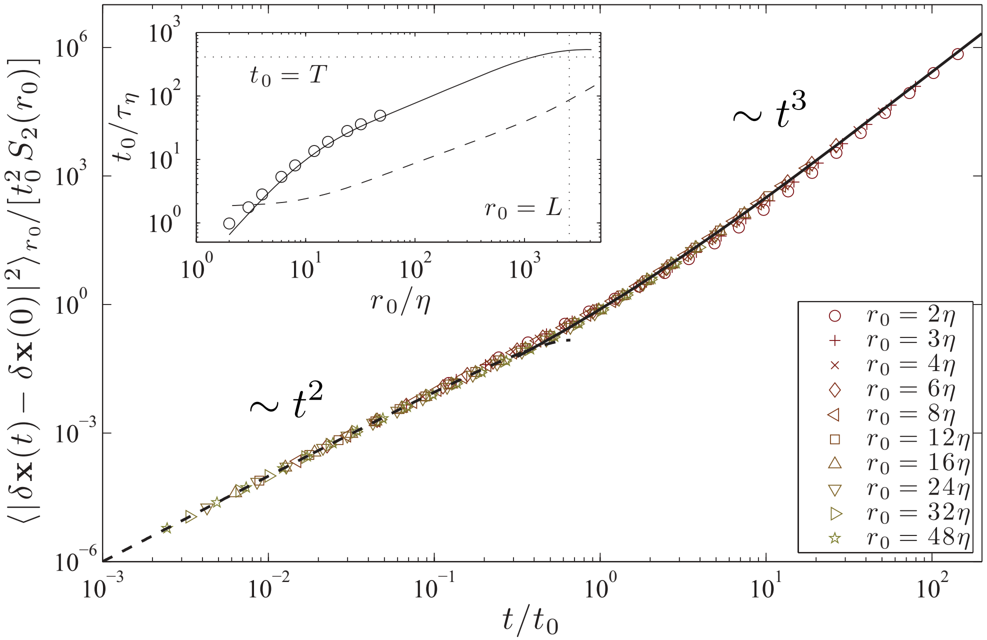


FIG. 1. (Color online) Time evolution of the mean-square separation for $R_\lambda = 730$ and various initial separations. The dashed line represents the behavior (2). The solid line is a fit to the Richardson regime (4) with $g = 0.52$ and $C = 1.6$. Inset: t_0 as a function of r_0 in dissipative-scale units. The solid line is an Eulerian average, the circles are Lagrangian measurements, and the dashed line is the turnover time $\tau(r_0)$.

R. Bitane, H. Homann, J. Bec PRE **86**, 045302(R) (2012)

Richardson diffusion: refined view

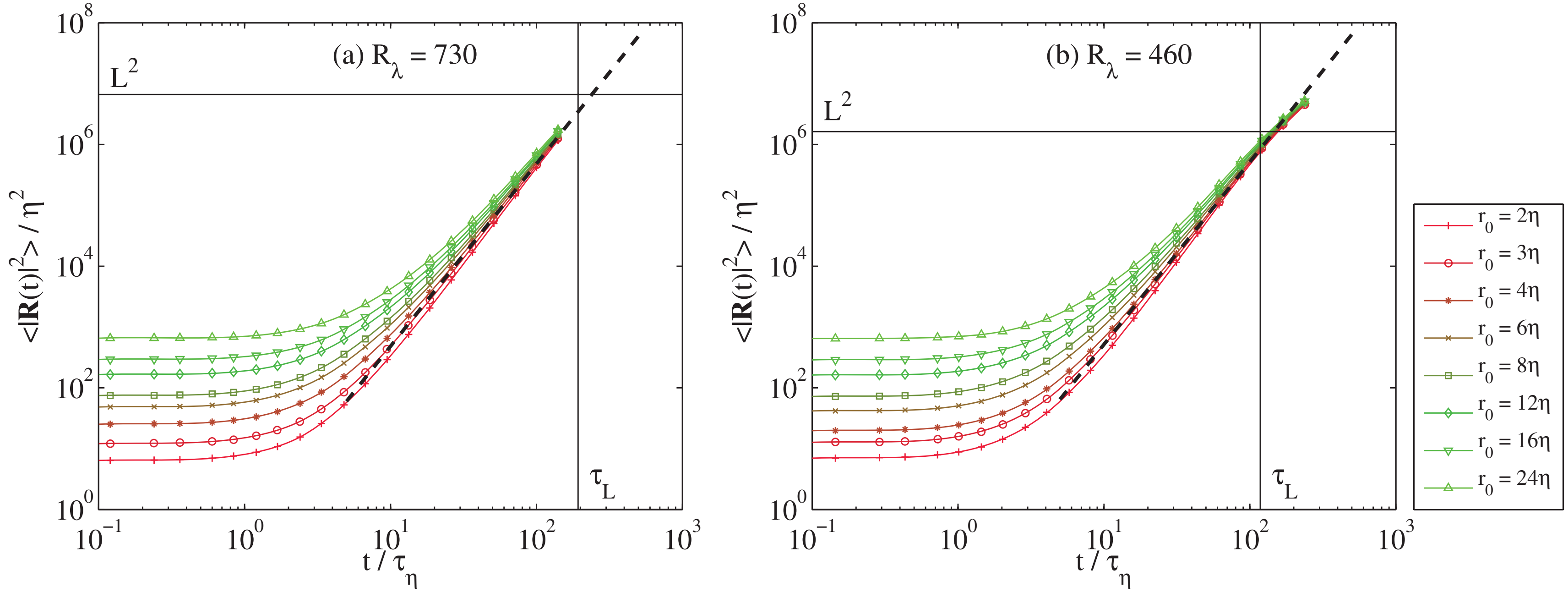


Figure 2. Time-evolution of the mean-squared distance for $R_\lambda = 730$ (a) and $R_\lambda = 460$ (b) for various initial separations r_0 as labeled. The horizontal and vertical solid lines represent the integral scale L and its associated turnover time τ_L, respectively. The dashed line corresponds to the explosive Richardson–Obukhov law (3) with $g = 0.52$.

R. Bitane, H. Homann, J. Bec JoT **14:2**, 23-45 (2013)

In the EXP only quite large initial separation were available and they could not be followed for a long time that's why only the ballistic (Batchelor) regime was found

Richardson diffusion: refined view

How good is Richardson PDF with respect to data?

$$P(R,t) \propto \frac{R^2}{\langle R^2(t)\rangle^{3/2}} \exp\left(-A\frac{R^{2/3}}{\langle R^2(t)\rangle^{1/3}}\right)$$

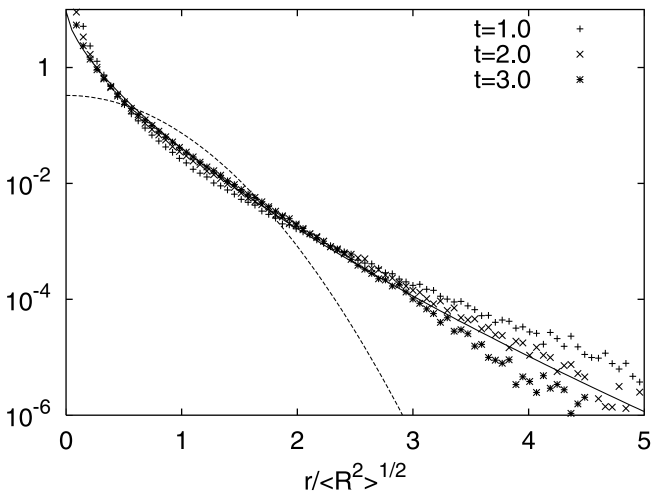


FIG. 2. Probability distribution function of relative separations at three different times. The continuous line is the Richardson prediction (3), and the dashed line is the Gaussian distribution proposed by Batchelor.

G. Boffetta & I. Sokolov PRL 88, 094501 (2002)

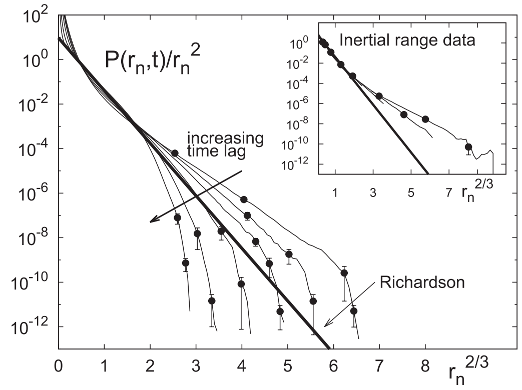


FIG. 3. Log-lin plot of $P(r_n, t)$ versus the rescaled variable r_n (see text) for $t = (20, 30, 40, 60, 90, 120)\tau_\eta$. The distribution $P(r_n, t)$ has been divided by a factor r_n^2 to highlight the large separation range. The Richardson prediction, Eq. (4), becomes time independent if rescaled in this way (solid curve). Inset: PDFs plotted only for separations r_n that, at time lag $t \in [10{:}120]\tau_\eta$, belong to the inertial subrange.

R. Scatamacchia, L. Biferale & F. Toschi PRL 109, 144501 (201

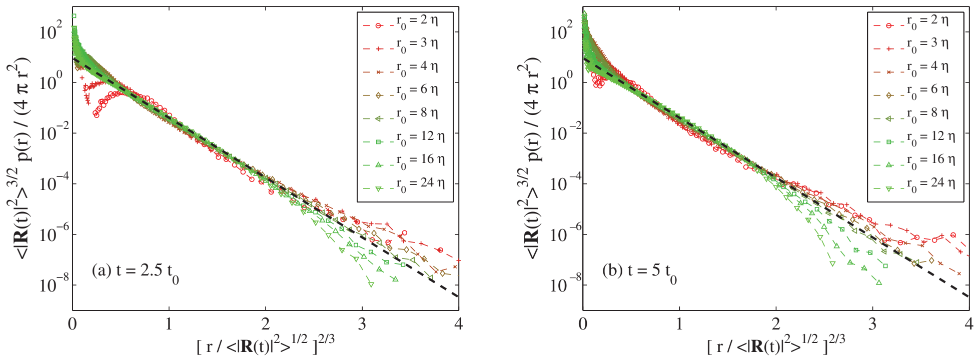
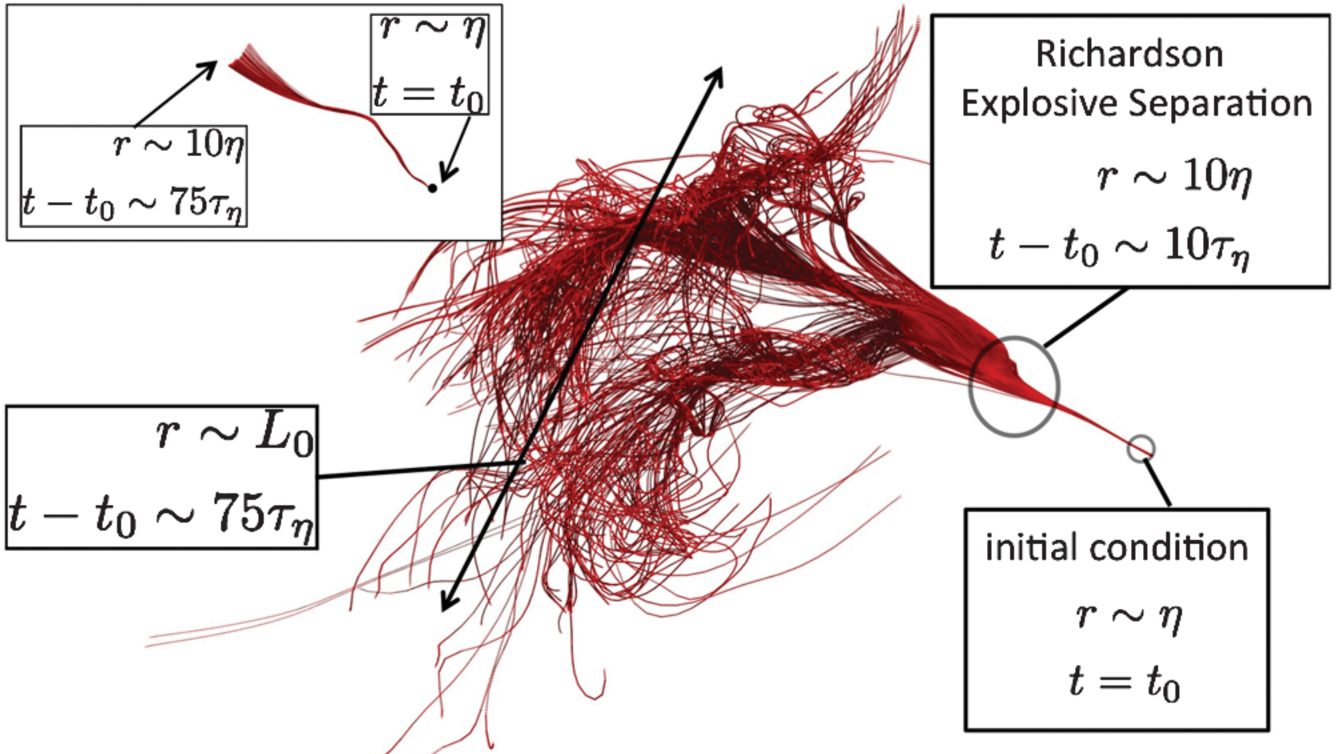


Figure 6. Probability density function of the distance r at time $t = 2.5\,t_0$ (a) and for $t = 5\,t_0$ (b) and for various values of the initial separation. We have here normalized it by $4\pi r^2$ and represented on a log y axis as a function of $r/\langle|\boldsymbol{R}(t)|^2\rangle^{1/2}$. With such a choice, Richardson's diffusive density distribution (2) appears as a straight line (represented here as a black dashed line).

R. Bitane, H. Homann, J. Bec JoT **14:2**, 23-45 (2013)

A source of difficulties



R. Scatamacchia, L. Biferale & F. Toschi PRL 109, 144501 (2012)

but see also R. Bitane, H. Homann, J. Bec JoT **14:2**, 23-45 (2013)

coexistence of pairs separating very fastly and pairs remaining close to each other for a long time
this makes very difficult to interpret statistics at fixed times

Richardson diffusion: are there effects of intermittency?

$$\langle R^p(t)\rangle \sim t^{\beta_p} \qquad \beta_p = \frac{3}{2}p$$ **dimensionally from Richardson**

What do we expect according to MF model?

$$\left\langle \frac{dR^p}{dt} \right\rangle = p\langle R^{p-1}\delta_R u\rangle$$

$$\delta_R u \sim \delta_L u \left(\frac{R}{L}\right)^h$$

$$\left\langle \frac{dR^p}{dt} \right\rangle = p\delta_L u L^{p-1} \int dh \left(\frac{R}{L}\right)^{p-1+h+3-D(h)}$$

MF relation between times and scales

$$t \sim \frac{R}{\delta_R u} \sim T_L \left(\frac{R}{L}\right)^{1-h} \qquad \frac{R}{L} \sim \left(\frac{t}{T_L}\right)^{\frac{1}{1-h}}$$

$$\left\langle \frac{dR^p}{dt} \right\rangle = p\frac{L^p}{T_L} \int dh \left(\frac{t}{T_L}\right)^{\frac{p-1+h+3-D(h)}{1-h}}$$

$$\beta_p = 1 + \min_h \left\{\frac{p-1+h+3-D(h)}{1-h}\right\} = \min_h \left\{\frac{p+3-D(h)}{1-h}\right\}$$

Richardson diffusion: are there effects of intermittency?

$$\langle R^p(t)\rangle \sim t^{\beta_p} \qquad \beta_p = \frac{3}{2}p$$ **dimensionally from Richardson**

$$\beta_p = 1 + \min_h \left\{ \frac{p - 1 + h + 3 - D(h)}{1-h} \right\} = \min_h \left\{ \frac{p + 3 - D(h)}{1-h} \right\}$$

p=2

$$\min_h \left\{ \frac{5 - D(h)}{1-h} \right\} = 3 \qquad \min_h \{3h + 3 - D(h)\} = 1$$

$$\beta_2 = 3$$

$$\langle R^2(t)\rangle = g\epsilon t^3$$

14 *L. Biferale, A. S. Lanotte, R. Scatamacchia, and F. Toschi*

JFM 757, 550-572 (2014)

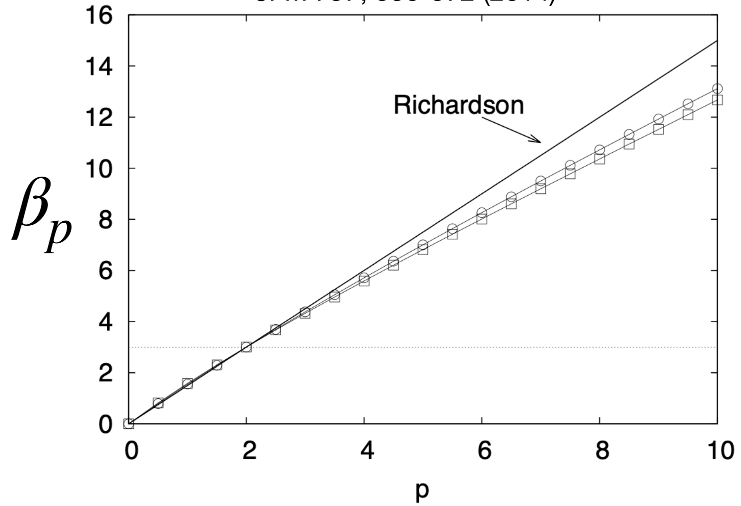


FIGURE 9. Multifractal exponents for pair separation statistics, derived from the scaling exponents of the Eulerian longitudinal structure functions, $D_L(h)$ (○), and from the scaling exponents of Eulerian transversal structure functions, $D_T(h)$ (□). The continuous line is the dimensional Richardson scaling, $\alpha(p) = 3p/2$.

Richardson diffusion: are there effects of intermittency?

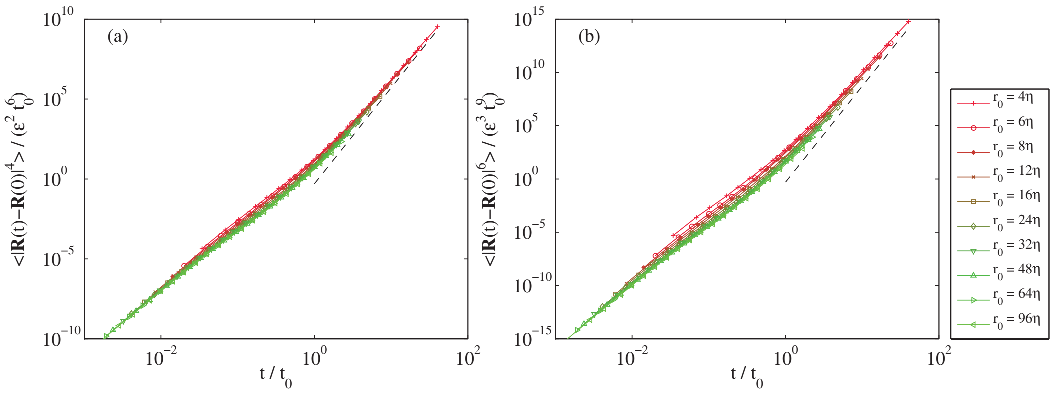


Figure 4. (a) Fourth-order moment $\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^4\rangle$ and (b) sixth-order moment $\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^6\rangle$ as function of t/t_0 for $R_\lambda = 730$. Both curves are normalized such that their expected long-time behavior is $\propto (t/t_0)^6$ and $\propto (t/t_0)^9$, respectively. The black dashed lines represent such behaviors.

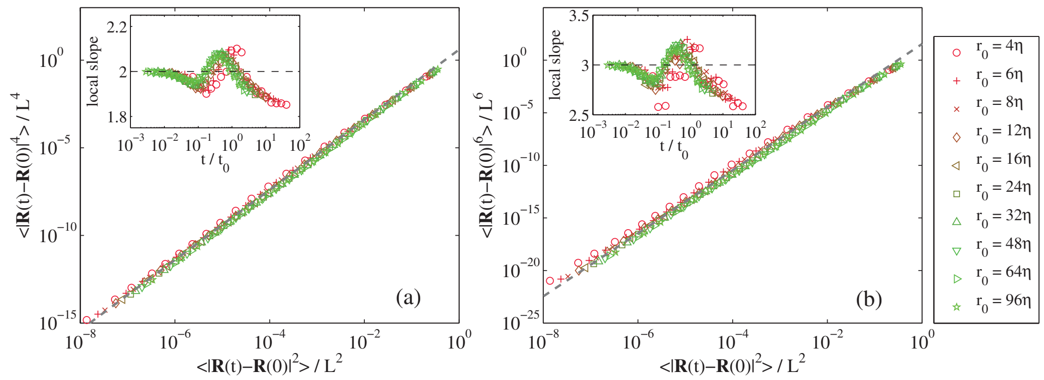
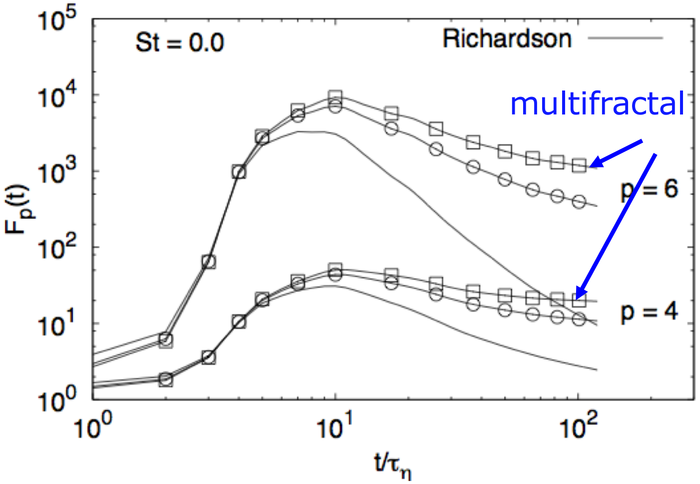


Figure 5. Fourth (a) and sixth (b) order moments of $|\boldsymbol{R}(t)-\boldsymbol{R}(0)|$ as a function of its second-order moment for $R_\lambda = 730$. The two gray dashed lines show a scale-invariant behavior, i.e., $\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^4\rangle \propto \langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^2\rangle^2$ and $\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^6\rangle \propto \langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^2\rangle^3$, respectively. The two insets show the associated local slopes, that is the logarithmic derivatives $\mathrm{d}\log\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^p\rangle/\mathrm{d}\log\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^2\rangle$, together with the normal scalings represented as dashed lines.

R. Bitane, H. Homann, J. Bec JoT **14:2**, 23-45 (2013)



$$\frac{\langle R^p(t)\rangle}{\langle R^2(t)\rangle^{\frac{\beta_p}{3}}}$$

L. Biferale, A.S Lanotte, R. Scatamacchia, & F. Toschi *JFM* 757 (2014): 550-572.

Richardson diffusion: intermittency & exit times

A possibility to mitigate the statistical effects of pairs that separates quickly or remains close for a long time is to look at the exit time statistics

Time for R(t) to pass from R to ρR with $\rho > 1$ this eliminates part of the problem with fixed time statistics and focus on the spatial scale avoiding averaging events which at the same time involve very different scales.

$$\mathcal{T}(R) = \{R(t) = R \quad \& \quad R(t + \mathcal{T}(R)) = \rho R\}$$

Then one can study their statistics

$$\mathcal{T}(R) \sim \frac{R}{\delta_R u} \sim T_L \left(\frac{R}{L}\right)^{1-h}$$

$$\left\langle \frac{1}{\mathcal{T}^p(R)} \right\rangle \sim \frac{1}{T_L^p} \int dh \left(\frac{R}{L}\right)^{-p+hp+3-D(h)} \sim \frac{1}{T_L^p} \left(\frac{R}{L}\right)^{\sigma_p}$$

$$\sigma_p = \zeta_p - p$$

K41+Richardson $\sigma_p = -\frac{2}{3}p$

G. Boffetta & I. Sokolov PRL 88, 094501 (2002)

G. Boffetta, A. Celani, A. Crisanti, A. Vulpiani, *PRE* **60**, 6734 (1999)

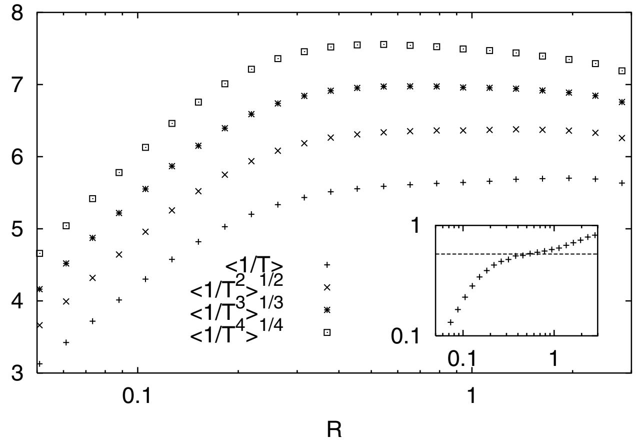


FIG. 3. First moments of the inverse doubling time $\langle [1/T(R)]^p \rangle$ compensated with Kolmogorov scaling $R^{-2p/3}$. Deviations from dimensional compensation are evident, in particular for $p = 4$. In the inset we plot the compensated mean doubling time according to (6) together with the estimate corresponding to $C_2 \simeq 0.55$.

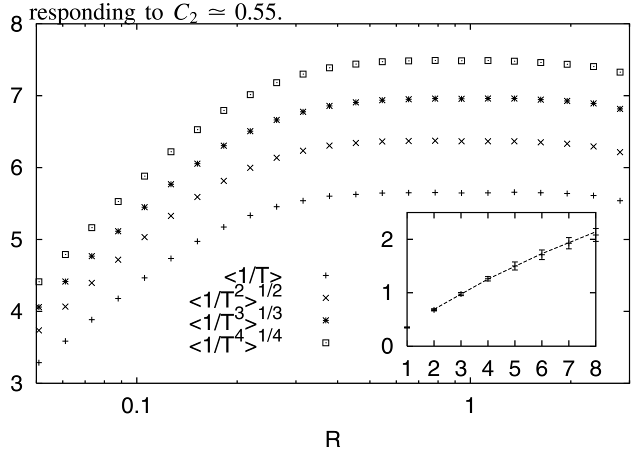


FIG. 4. First moments of the inverse doubling time $\langle [1/T(R)]^p \rangle$ compensated with best fit exponent β_p. Observe the improvement in the compensation with respect to Fig. 3. In the inset we plot the structure function exponents estimated from $\zeta_p = p + \beta_p$. The dashed line represents the Eulerian exponents obtained by ESS analysis.

Richardson diffusion & role of intermittency

Not completely settled

VOLUME 88, NUMBER 9 PHYSICAL REVIEW LETTERS 4 MARCH 2002

Relative Dispersion in Fully Developed Turbulence: The Richardson's Law and Intermittency Corrections

G. Boffetta[1] and I. M. Sokolov[2]

PRL **109,** 144501 (2012) PHYSICAL REVIEW LETTERS week ending 5 OCTOBER 2012

Extreme Events in the Dispersions of Two Neighboring Particles Under the Influence of Fluid Turbulence

R. Scatamacchia,[1,2] L. Biferale,[1] and F. Toschi[3]

PHYSICAL REVIEW E VOLUME 60, NUMBER 6 DECEMBER 1999

Pair dispersion in synthetic fully developed turbulence

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and Istituto Nazionale di Fisica della Materia, Unità di Roma I, Rome, Italy
(Received 1 June 1999)

PHYSICAL REVIEW LETTERS **131,** 064001 (2023)

Spontaneous Stochasticity in the Presence of Intermittency

André Luís Peixoto Considera[1,2] and Simon Thalabard[3]

[redacted]

PHYSICAL REVIEW E **86**, 045302(R) (2012)

Time scales of turbulent relative dispersion

Rehab Bitane, Holger Homann, and Jérémie Bec
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(Received 29 June 2012; published 31 October 2012)

Journal of Turbulence, 2013
Vol. 14, No. 2, 23–45, http://dx.doi.org/10.1080/14685248.2013.766747
Taylor & Francis
Taylor & Francis Group

Geometry and violent events in turbulent pair dispersion

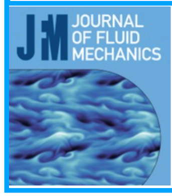
Rehab Bitane, Holger Homann and Jérémie Bec*

J. Fluid Mech. (2014), *vol.* 755, R4, doi:10.1017/jfm.2014.445

JFM RAPIDS
journals.cambridge.org/rapids

Turbulent pair dispersion as a continuous-time random walk

Simon Thalabard[1], Giorgio Krstulovic[1] and Jérémie Bec[1,†]



Turbulent pair dispersion as a ballistic cascade phenomenology

Published online by Cambridge University Press: **08 May 2015**

Mickael Bourgoin

Journal of Fluid Mechanics 772 (2015): 678-704.

Richardson dispersion & irreversibility

$$\langle|\boldsymbol{R}(t)-\boldsymbol{R}_0|^2\rangle = \langle(\delta_{R_0}\boldsymbol{u}\cdot\delta_{R_0}\boldsymbol{u})\rangle t^2 + \underbrace{\langle\delta_{R_0}\boldsymbol{u}\cdot\delta_{R_0}\boldsymbol{a}\rangle}_{\approx -2\epsilon} t^3 + o(t^3)$$

J. Mann, S. Ott, & JS Andersen (1999)

R.J. Hill JoT 7, N43 (2006)

$$\langle|\delta\boldsymbol{R}(t)|^2\rangle = pos\ t^2 + neg\ t^3$$

$$\langle|\delta\boldsymbol{R}(-t)|^2\rangle - \langle|\delta\boldsymbol{R}(t)|^2\rangle = -2 * neg\ |t|^3 = 4\epsilon t^3$$

Backward separation is faster than forward separation

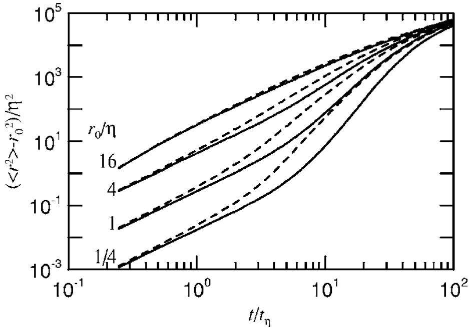


FIG. 4. Forwards relative dispersion (solid lines) and backwards relative dispersion (dashed lines) for initial separations $r_0/\eta=\frac{1}{4}$, 1, 4, and 16 as labeled for DNS calculations at $\mathrm{Re}_\lambda=38$.

B. L. Sawford et al
PoF 17, 095109 (2005)

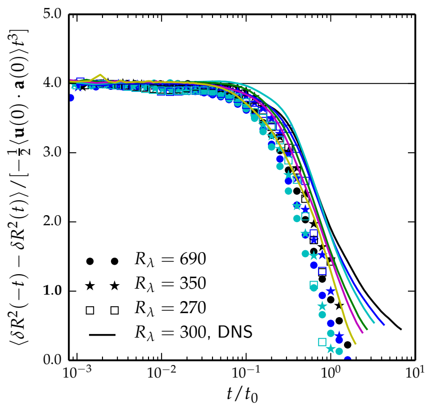


FIG. 1. (color online). The difference between the backward and forward mean squared relative separation, $\langle\delta\mathbf{R}(-t)^2 - \delta\mathbf{R}(t)^2\rangle$, compensated using Eq. (5). The symbols correspond to experiments: circles for $R_\lambda = 690$ ($R_0/\eta = 267, 333, 400$), stars for $R_\lambda = 350$ ($R_0/\eta = 152, 182, 212$), and squares for $R_\lambda = 270$ ($R_0/\eta = 95, 114, 133$). The lines correspond to DNS at $R_\lambda = 300$ ($R_0/\eta = 19, 38, 58, 77, 92, 123$).

J Jucha et al. *PRL* 113.5 (2014): 054501.

105101-8 Buaria, Sawford, and Yeung Phys. Fluids **27**, 105101 (2015)

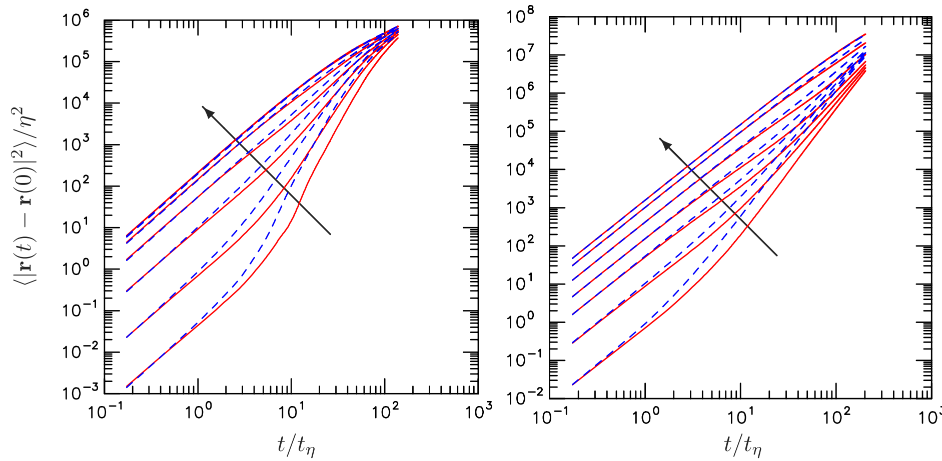


FIG. 2. Mean-squared relative displacement as a function of forward time (solid lines, in red) and backward time (dashed lines, in blue) at R_λ 140 (left) and 1000 (right), scaled by Kolmogorov variables, for different initial separations. Arrows indicate direction of increasing $\tilde{r}_0$, in logarithmically spaced intervals: $\tilde{r}_0/\eta = 1/4$, 1, 4, 16, 64, 256, and 1024 for R_λ 140; $\tilde{r}_0/\eta = 1$, 4, 16, 64, 256, 1024, and 4096 for R_λ 1000. (Results at $\tilde{r}_0/\eta = 1/4$ for R_λ 1000 are not shown since they are not well sampled.)

Richardson dispersion on the field

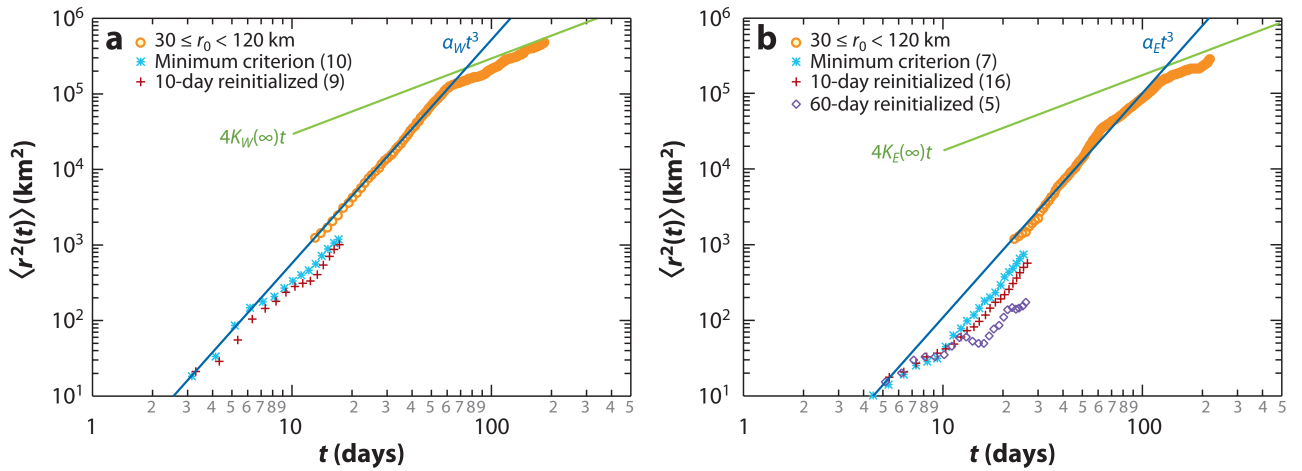


Figure 6

Mean square separation distance versus time obtained from time-shifted float data during the TOPOGULF experiment: (*a*) western floats and (*b*) eastern floats. $r_0 < 7.5$ km for the 10-day (*crosses*) and 60-day (*open diamonds*) reinitialized time series. Asterisks give the corresponding evolution for portions selected with the minimum separation criterion. This consists of finding the minimum separation distance over the entire lifetime of the pair and recording the evolution for 120 days thereafter. The remaining record is then searched for a new minimum and the process is repeated. Numbers of pairs are indicated within parentheses. α_W and α_E are numerical coefficients, and $K_E(\infty)$ and $K_W(\infty)$ are the diffusion coefficients in the diffusive limit. Figure adapted from Ollitrault et al. 2005.

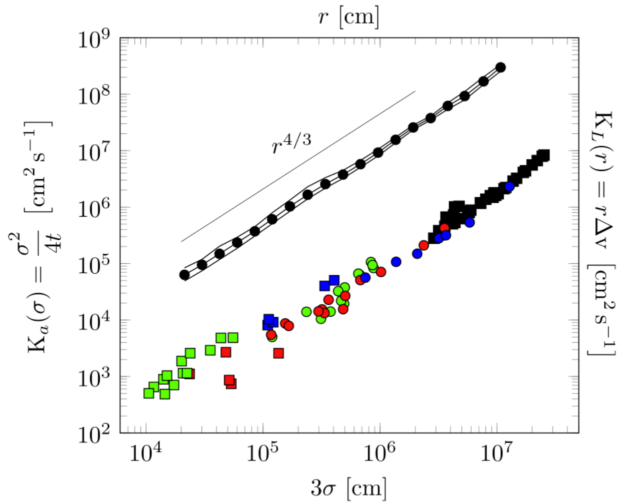


Fig. 6. Scale-dependent relative diffusivities. Lower left axes: Tracer-based diffusivity estimates based on fitting ellipses. The solid red, green, and blue symbols show Okubo (36) estimates of $K_a(3\sigma) = \sigma^2/4t$. Corresponding estimates from the $\mathcal{S}1$ drifter data are shown by solid black squares. Upper right axes: Scale-dependent mixing length diffusivities, $K_L(r) = r\Delta v(r)$, observed in $\mathcal{S}1$ launch plotted with uncertainty estimates in solid black lines and filled black circles. Richardson–Obukhov scaling law, $K_L(r) \sim r^{4/3}$ is indicated.

Ollitrault M, Gabillet C, de Verdie`re AC. 2005. Open ocean regimes of relative dispersion. *J. Fluid Mech*. 533:381–407

Poje, Andrew C., et al. "Submesoscale dispersion in the vicinity of the Deepwater Horizon spill." *PNAS* 111.35 (2014): 12693-12698.

Two-Particle Dispersion in Isotropic Turbulent Flows

Juan P.L.C. Salazar and Lance R. Collins Annu. Rev. Fluid Mech. 2009. 41:405–32

Richardson diffusion: why so difficult?

Which are the difficulties in studying Lagrangian dispersion ?

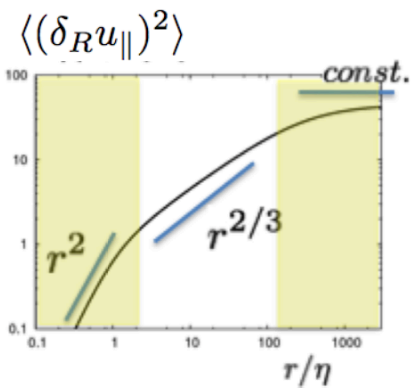
1 need to accurately know the spatial statistics of the flow velocity along Lagrangian paths *high-resolution, high frequency*

2 need scale separation to disentangle different dispersion regimes: exponential, ballistic, turbulent,..

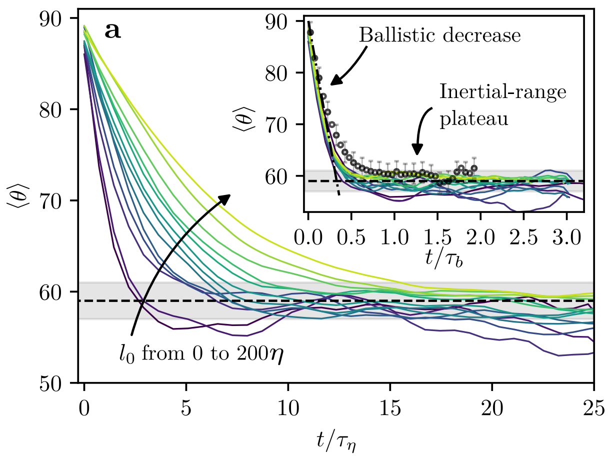
3 need to have high statistical accuracy : *long records* along *many Lagrangian paths*

4 need to limit the impact of inhomogeneities (walls, borders), unsteadiness, anisotropies, stratification

Finite Re effects are very severe



Persistency of correlations



Breakdown of the Lagrangian flow

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Re^d$$

be f a smooth velocity field

if f is continuous with the Lipschitz condition
(essentially if f is differentiable)

$$||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|| \leq K||\boldsymbol{x} - \boldsymbol{y}||$$

The solution exists and is unique

$$p(\boldsymbol{x}, t|\boldsymbol{y}, s) = \delta(\boldsymbol{x} - \boldsymbol{x}(t; \boldsymbol{y}, s))$$

$$\delta\boldsymbol{x}(t) = \boldsymbol{x}'(t) - \boldsymbol{x}(t)$$

$$\delta\boldsymbol{x}(0) = \boldsymbol{x}'(0) - \boldsymbol{x}(0) = \varepsilon$$

in the presence of chaos $|\delta\boldsymbol{x}(t)| = \epsilon e^{\lambda_1 t}$ **yet** $\lim_{\epsilon \to 0} |\delta\boldsymbol{x}(t)| = 0$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) + \sqrt{2D_0}\boldsymbol{\eta}(t)$$

$$\partial_t p(\boldsymbol{x}, t) + \boldsymbol{\nabla} \boldsymbol{f}(\boldsymbol{x}) p(\boldsymbol{x}, t) - D_0 \Delta p(\boldsymbol{x}, t) = 0$$

$$p(\boldsymbol{x}, s) = \delta(\boldsymbol{x} - \boldsymbol{y})$$

$$\lim_{D_0 \to 0} p(\boldsymbol{x}, t|\boldsymbol{y}, s) = \delta(\boldsymbol{x} - \boldsymbol{x}(t; \boldsymbol{y}, s))$$

Breakdown of the Lagrangian flow

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x})$$ **be u a non smooth incompressible velocity field e.g.** $\delta_R \boldsymbol{u} \sim R^{1/3}$

1d example

$$\frac{dR}{dt} = \delta_R u = R^{1/3}$$

$$\partial_t p(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} p(\boldsymbol{x},t) - D_0 \Delta p(\boldsymbol{x},t) = 0$$

$$R(s) = \epsilon$$

$$p(\boldsymbol{x},s) = \delta(\boldsymbol{x}-\boldsymbol{y})$$

$$R(t) = \left[\frac{2}{3}(t-s) + \epsilon^{2/3}\right]^{3/2}$$

$$\lim_{D_0 \to 0} p(\boldsymbol{x},t|\boldsymbol{y},s) = \mathcal{P}(\boldsymbol{x},t)$$

weak solution of

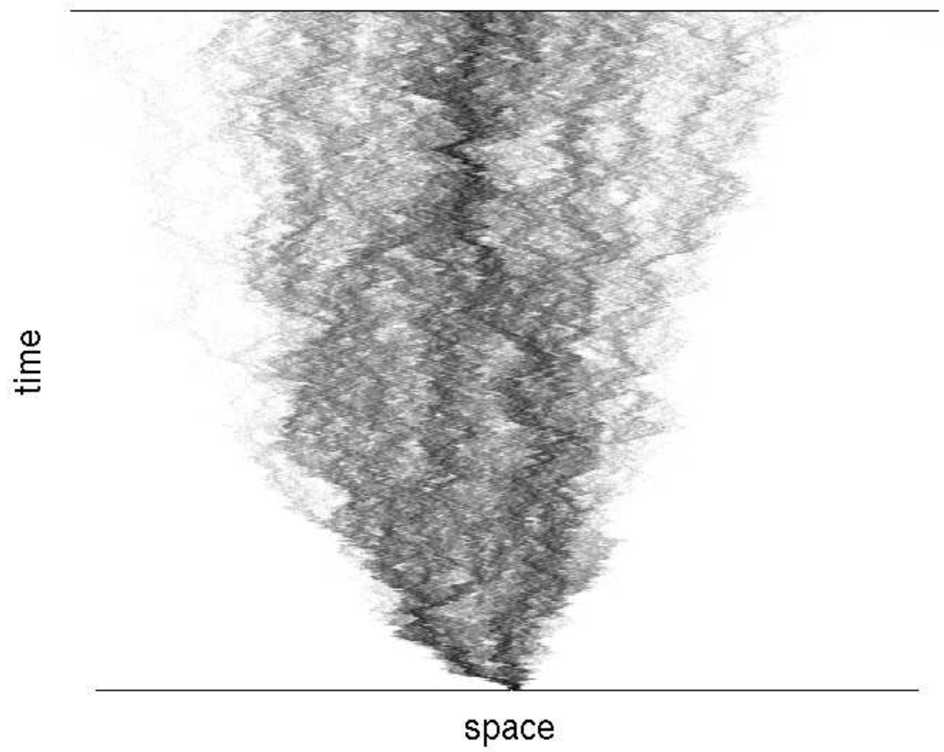
$$\partial_t \mathcal{P}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} \mathcal{P}(\boldsymbol{x},t) = 0$$

if $\epsilon = 0$ **or** $t-s \gg \epsilon^{2/3}$

$$R(t) = \left[\frac{2}{3}(t-s)\right]^{3/2}$$

but for $\epsilon = 0$ **also** $R(t) = 0$ **is a solution**

the Lagrangian flow breaks a trajectory is no more labeled by its initial condition



Breakdown of the Lagrangian flow

We shall come back to this explosive separation when discussing the transport of substances in turbulent flows

A clean framework for (not only) Richardson dispersion

The Kraichnan model (R. H. Kraichnan, PoF 11,11, 945 (1968))

$$\dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r}, t) + \sqrt{2\kappa}\boldsymbol{\eta}$$

$\boldsymbol{v}(\boldsymbol{r}, t)$ **zero mean Gaussian velocity field with correlation** $\langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t')\rangle = 2\,\delta(t-t')D^{ij}(\mathbf{r}-\mathbf{r}')$

$$D^{ij}(\mathbf{r}) = D_0\delta^{ij} - \frac{1}{2}d^{ij}(\mathbf{r}) \qquad \lim_{\substack{\eta\to 0\\ L\to\infty}} d^{ij}(\mathbf{r}) = D_1 r^{\xi}\left((d-1+\xi)\,\delta^{ij} - \xi\frac{r^i r^j}{r^2}\right)$$

rough **smooth**

$$0 \leqslant \xi \leqslant 2$$

NB: is an incompressible ensemble of velocities

$$\sum_i \partial_{r_i}\langle v^i(\boldsymbol{r},t)v^j(\boldsymbol{r}',t')\rangle = \sum_j \partial_{r'_j}\langle v^i(\boldsymbol{r},t)v^j(\boldsymbol{r}',t')\rangle = 0 \;\Rightarrow\; \sum_i \partial_{r_i} D^{ij}(\boldsymbol{r}-\boldsymbol{r}') = \sum_j \partial_{r_j} D^{ij}(\boldsymbol{r}-\boldsymbol{r}') = 0$$

the delta correlation makes the dynamics of particles reversible

B. Shraiman & E. D. Siggia. N*ature* 405, 639 (2000) G. Falkovich, K. Gawędzki, M. Vergassola. RMP 73, 913 (2001)

A clean framework for (not only) Richardson dispersion

$$\langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t')\rangle = 2\,\delta(t-t')D^{ij}(\mathbf{r}-\mathbf{r}')$$

$$D^{ij}(\mathbf{r}) = D_0\delta^{ij} - \frac{1}{2}d^{ij}(\mathbf{r}) \qquad \lim_{\substack{\eta\to 0\\ L\to\infty}} d^{ij}(\mathbf{r}) = D_1 r^{\xi}\left((d-1+\xi)\,\delta^{ij} - \xi\frac{r^i r^j}{r^2}\right)$$

Two particles joint probability at two different times

$$\langle p(\mathbf{r}_1,s;\mathbf{R}_1,t|\mathbf{v})p(\mathbf{r}_2,s;\mathbf{R}_2,t|\mathbf{v})\rangle_{\mathbf{v}} \equiv \mathcal{P}_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{R}_1,\mathbf{R}_2;t-s)$$

obeys the equation

$$(\partial_t - \mathcal{M}_2)\mathcal{P}_2 = \delta(t-s)\,\delta(\mathbf{R}_1-\mathbf{r}_1)\,\delta(\mathbf{R}_2-\mathbf{r}_2)$$

$$\mathcal{M}_2 = \sum_{n,n'=1}^{2} D^{ij}(\mathbf{r}_n - \mathbf{r}_{n'})\nabla_{r_n^i}\nabla_{r_{n'}^j} + \kappa\sum_{n=1}^{2}\nabla_{\mathbf{r}_n}^2$$

relative motion $r = r_1 - r_2$ **and** $R = R_1 - R_2$

$$(\partial_t - M)\mathcal{P}(r;R,t) = \delta(t)\delta(R-r)$$

$$M = \frac{1}{r^{d-1}}\partial_r[(d-1)D_1 r^{d-1+\xi} + 2\kappa r^{d-1}]\partial_r$$

$\xi = 2$
$$\lim_{\substack{r\to 0\\ \kappa\to 0}} \mathcal{P}(r;R;t) = \delta(R)$$

$0 \le \xi < 2$
$$\lim_{\substack{r\to 0\\ \kappa\to 0}} \mathcal{P}(r;R;t) \propto \frac{R^{d-1}}{|t|^{d/(2-\xi)}}\exp\left[-\mathrm{const}\times\frac{R^{2-\xi}}{|t|}\right]$$

$$\langle R^2(t)\rangle \propto t^{2/(2-\xi)}$$

B. Shraiman & E. D. Siggia. N*ature* 405, 639 (2000) G. Falkovich, K. Gawędzki, M. Vergassola. RMP 73, 913 (2001)

Multi-particle separation

Multi-particle separation

N particles $\dot{\boldsymbol{r}}_i = \boldsymbol{v}(\boldsymbol{r}_i, t) + \sqrt{2\kappa}\boldsymbol{\eta}_i \qquad i = 1, \ldots, N$

Multiparticle propagator

$$\left\langle \prod_{n=1}^{N} p(\mathbf{r}_n, s; \mathbf{R}_n, t | \mathbf{v}) \right\rangle_{\mathbf{v}} \equiv \mathcal{P}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}}; t-s)$$

The relative motion of N particles may be described by the versions of the joint PDF's (65) averaged over rigid translations:

$$\tilde{\mathcal{P}}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}}; t) = \int \mathcal{P}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}} + \underline{\boldsymbol{\rho}}; t)\, d\boldsymbol{\rho}, \tag{66}$$

where $\underline{\boldsymbol{\rho}} = (\boldsymbol{\rho}, \ldots, \boldsymbol{\rho})$. The PDF in Eq. (66) describes the distribution of the particle separations $\mathbf{R}_{nm} = \mathbf{R}_n - \mathbf{R}_m$ or of the relative positions $\underline{\tilde{\mathbf{R}}} = (\mathbf{R}_1 - \bar{\mathbf{R}}, \ldots, \mathbf{R}_N - \bar{\mathbf{R}})$.

N=2 we saw that the separation R grows as a power law

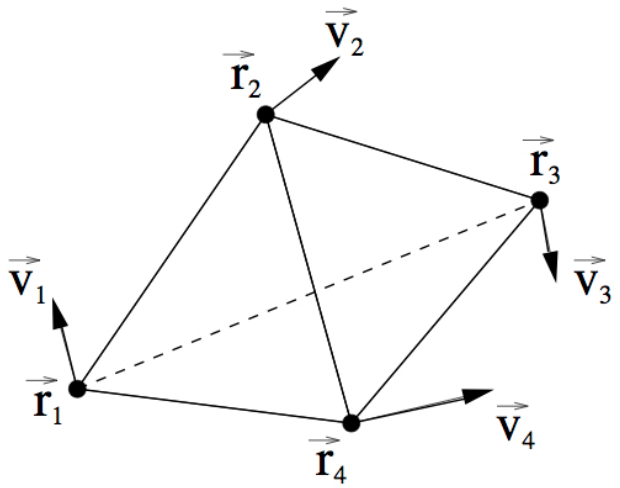
$R \sim t^{3/2}$ turbulence

$R \sim t^{1/(2-\xi)}$ Kraichnan

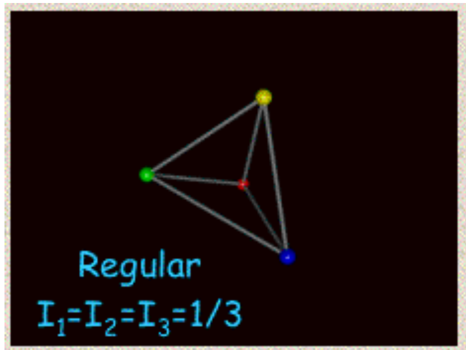
When N>2 besides the evolution of the size we also have evolution of the shape to account for

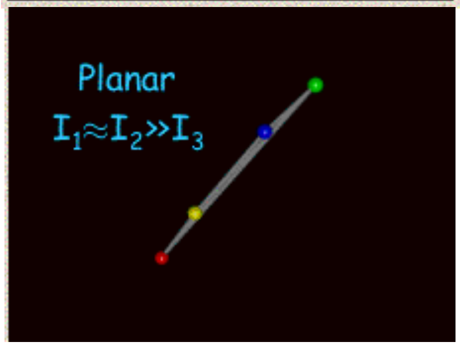
Multi-particle separation: e.g. N=4

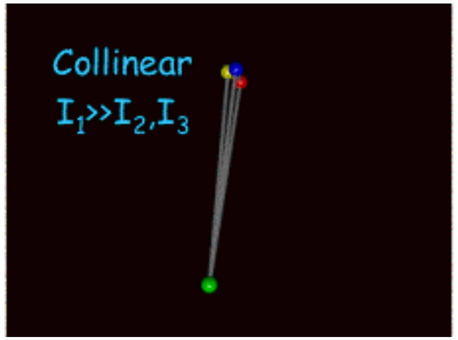
(Chertkov, Pumir, Shraiman 1999)



$$\rho_0 = (\mathbf{x_1} + \mathbf{x_2} + \mathbf{x_3} + \mathbf{x_4})/2$$
$$\rho_1 = (\mathbf{x_1} - \mathbf{x_2})/\sqrt{2}$$
$$\rho_2 = (\mathbf{2x_3} - \mathbf{x_2} - \mathbf{x_1})/\sqrt{6}$$
$$\rho_3 = (\mathbf{3x_4} - \mathbf{x_3} - \mathbf{x_2} - \mathbf{x_1})/\sqrt{12}$$







It can be studied in terms of the eingenvalues "g" of the inertia matrix I = $\rho\rho^T$ built from vectors ρ_1, ρ_2, ρ_3 in the set of coordinates independent of centre of mass ρ_0

$$\mathcal{I} = \rho\rho^T \Rightarrow \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

- $r = \sqrt{g_1 + g_2 + g_3} = \sqrt{\frac{1}{8}\sum_{ij} |\boldsymbol{x}_i - \boldsymbol{x}_j|}$ *tetrad size*
- $\boxed{I_i \equiv \frac{g_i}{r^2}, \quad I_1 + I_2 + I_3 \equiv 1}$ *tetrad shape*

$I_1 = I_2 = I_3 = 1/3$ *regular*
$I_3 = 0$ *coplanar*
$I_2 = I_3 = 0$ *collinear*

Multi-particle separation: N=4

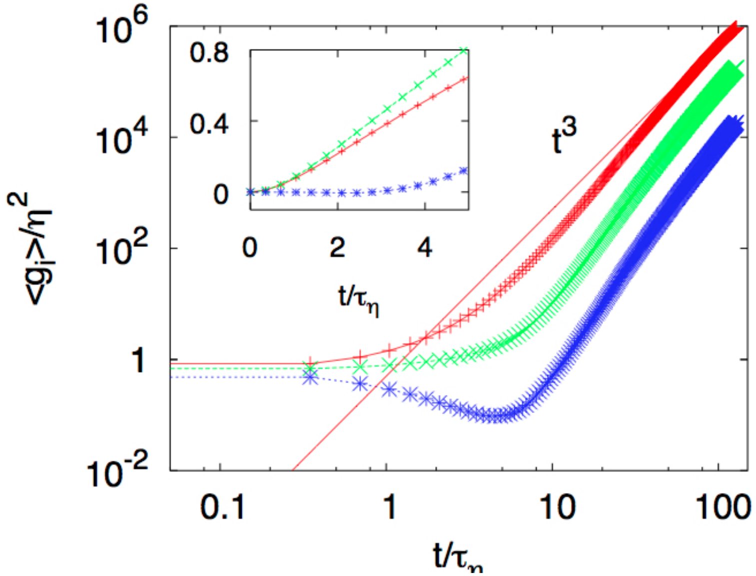


FIG. 1. Evolution of the mean eigenvalues g_1 (+), g_2 (×) and g_3 (*) of the moment of inertia matrix $\boldsymbol{I}=\boldsymbol{\rho\rho}^T$. The line represents the dimensional scaling t^3. In the inset, from top to bottom: evolution at small times of $\langle \ln A(t)\rangle$ (surface), $\langle \ln R(t)\rangle$ (distance), $\langle \ln V(t)\rangle$ (volume). The linear slopes of the three curves in the range of times $\tau_\eta < t < 3\tau_\eta$ yield $\lambda_1+\lambda_2$, λ_1, and $\lambda_1+\lambda_2+\lambda_3$, respectively.

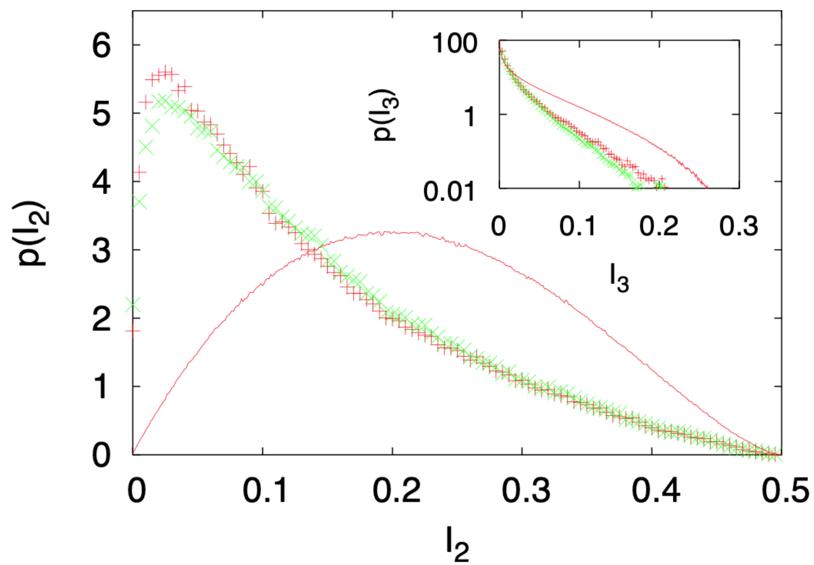


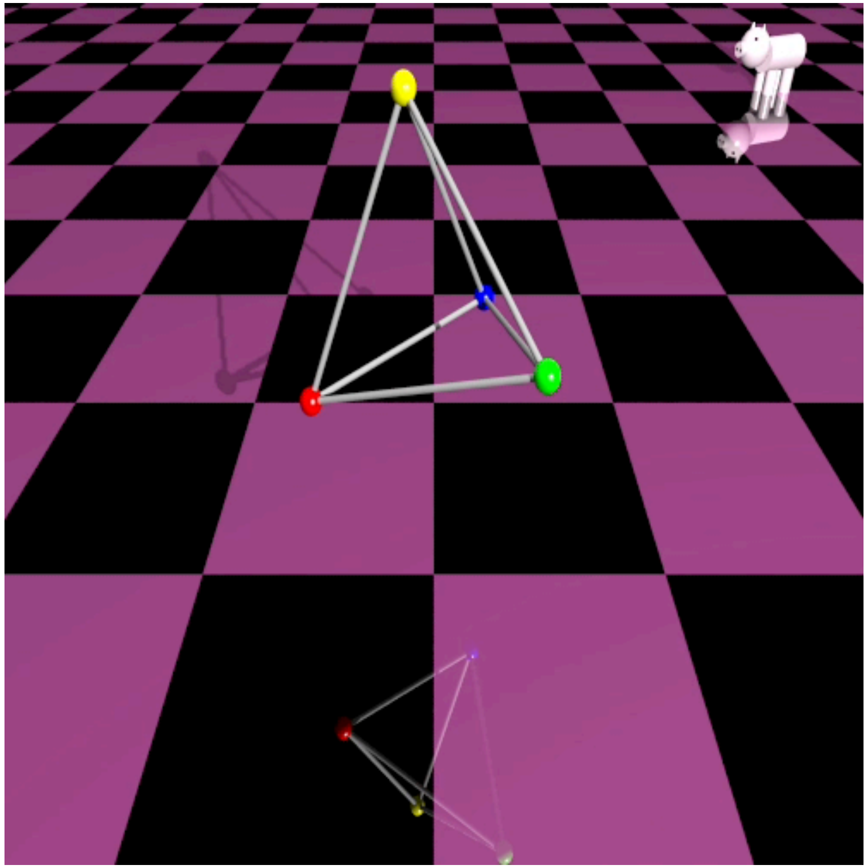
FIG. 4: Probability density function of shape indices I_2 and I_3 (inset) at times $t = 35\tau_\eta$ (+) and $t = 63\tau_\eta$ (×). The full lines are the pdfs for independent, Gaussian distributed particle positions.

$$\langle I_1\rangle = 0.854 \quad \langle I_2\rangle = 0.135 \quad \langle I_3\rangle = 0.011$$

In general it was found $I_2 \ll I_1$ and $I_3 \ll I_2$ denoting preference for elongated and planar geometries

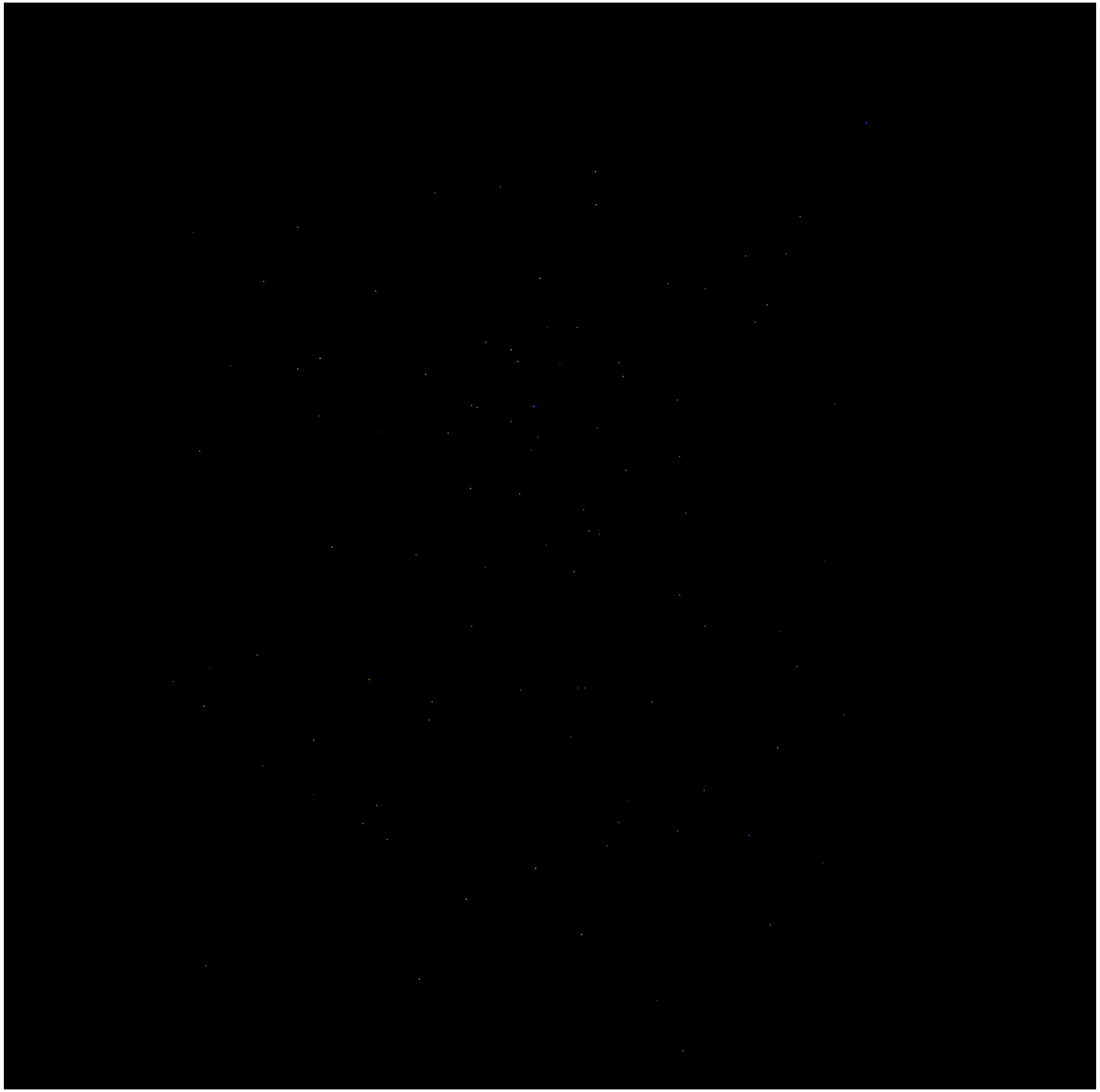
L. Biferale; G. Boffetta; A. Celani; B. J. Devenish; A. Lanotte; F. Toschi PoF 17, 111701 (2005)
see also A. Pumir, B.I. Shraiman, M. Chertkov PRL 85, 5324 (2000) for a previous study at lower Re

Time evolution of tetrads in 3D turbulence

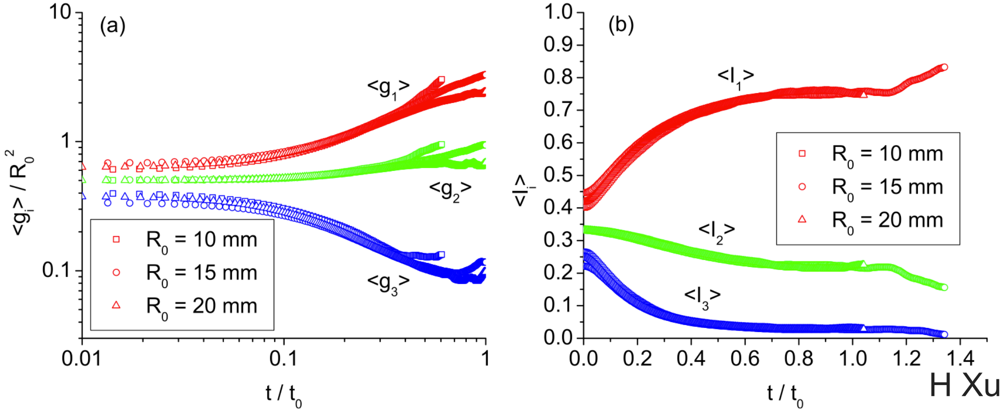


Courtesy of Guido Boffetta

L. Biferale; G. Boffetta; A. Celani; B. J. Devenish; A. Lanotte; F. Toschi PoF 17, 111701 (2005)



Multi-particle separation: N=4



EXP are in qualitative agreement with DNS

H Xu,, N T. Ouellette, E Bodenschatz. *New Journal of Physics* 10, no. 1 (2008): 013012.

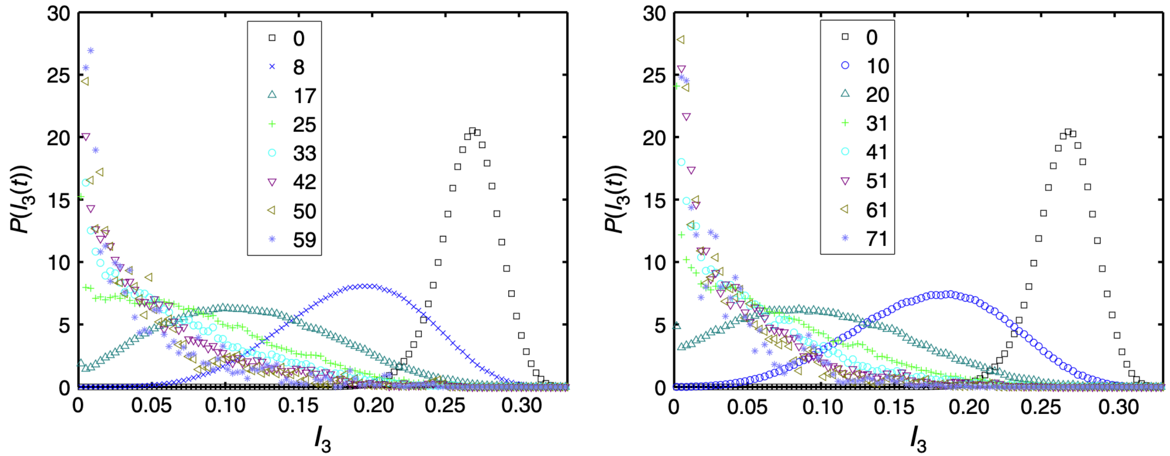


Figure 5. The evolution of the PDF of the shape factor $\langle I_3 \rangle$ for tetrads with an initial size of $R_0 = 20$ mm. The legends show the time (in units of τ_η) at which the PDFs are measured. Similar changes are observed for tetrads with initial size of 10 and 15 mm. (a) $R_\lambda = 690$ ($\eta = 30\,\mu$m); (b) $R_\lambda = 815$ ($\eta = 23\,\mu$m).

But:

1. **no evidence of t^3 size growth**
2. **difficulty to have long time information due to finite volume effects in the experiment**
3. **size of initial tetraedon is quite large**

Multi-particle separation: Lagrangian average

Let us reconsider the Kraichnan model $\dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r},t) + \sqrt{2\kappa}\boldsymbol{\eta}$ $\quad \langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t')\rangle = 2\,\delta(t-t')D^{ij}(\mathbf{r}-\mathbf{r}')$

$$D^{ij}(\mathbf{r}) = D_0\delta^{ij} - \frac{1}{2}d^{ij}(\mathbf{r}) \qquad \lim_{\substack{\eta\to 0\\ L\to\infty}} d^{ij}(\mathbf{r}) = D_1 r^{\xi}\left((d-1+\xi)\,\delta^{ij} - \xi\frac{r^i r^j}{r^2}\right)$$

rough **smooth**

$0 \leqslant \xi \leqslant 2$

For $\xi = 0$ (i.e. rough flow) particles undergo independent Brownian motions in the limit $\kappa \to 0$

$$\boldsymbol{R}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j \qquad \langle R_{ij}^2(t)\rangle = r_{ij}^2 + D_1 t \qquad r_{ij} = R_{ij}(0)$$

Now consider N=4 particles and functions, $f(\boldsymbol{R}_{12}, \boldsymbol{R}_{3,4})$ **, of their relative separation**

e.g. $f(\boldsymbol{R}_{12}, \boldsymbol{R}_{3,4}) = R_{12}^2 - R_{3,4}^2$

Clearly if we average along the Lagrangian trajectories $\langle f\rangle(t) = \langle f(\boldsymbol{R}_{12}(t), \boldsymbol{R}_{34}(t))\rangle = \langle R_{12}^2(t) - R_{34}^2(t)\rangle = r_{12}^2 - r_{34}^2 = const$

the same happens to $f(\boldsymbol{R}_{12}, \boldsymbol{R}_{34}) = 2(d+2)R_{12}^2 R_{34}^2 - d(R_{12}^4 + R_{34}^4)$

An example for smooth flows $\xi = 2$ $f(\boldsymbol{R}_{12}, \boldsymbol{R}_{34}) = R_{12}^2/R_{34}^2$ **indeed** $R_{ij} = r_{ij}e^{\lambda t}$

When this happens such functions are called zero modes

How does it work?

Multi-particle separation & zero modes

Be $\underline{\boldsymbol{R}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$ **and** $\underline{\tilde{R}}$ **the restriction to variables only depending on relative separations of N particles**

Lagrangian average $\langle f \rangle(\underline{\tilde{\boldsymbol{r}}}, t) = \langle f(\underline{\tilde{\boldsymbol{R}}}(t; \underline{\tilde{\boldsymbol{r}}})) \rangle = \int f(\underline{\tilde{\boldsymbol{R}}}') \, \tilde{\mathcal{P}}(\underline{\tilde{\boldsymbol{r}}}; \underline{\tilde{\boldsymbol{R}}}'; t) d\underline{\tilde{\boldsymbol{R}}}'$

$$\frac{d\langle f \rangle(\underline{\tilde{\boldsymbol{r}}}, t)}{dt} = \int f(\underline{\tilde{\boldsymbol{R}}}') \, \partial_t \tilde{\mathcal{P}}_N(\underline{\tilde{\boldsymbol{r}}}; \underline{\tilde{\boldsymbol{R}}}'; t)) d\underline{\tilde{\boldsymbol{R}}}' = \int f(\underline{\tilde{\boldsymbol{R}}}') \, \tilde{\mathcal{M}}_N \tilde{\mathcal{P}}_N(\underline{\tilde{\boldsymbol{r}}}; \underline{\tilde{\boldsymbol{R}}}'; t)) d\underline{\tilde{\boldsymbol{R}}}'$$

Kraichnan Model

$$\partial_t \tilde{\mathcal{P}}_N = \tilde{\mathcal{M}}_N \tilde{\mathcal{P}}_N \qquad \tilde{\mathcal{M}}_N = -\sum_{n<m} [d^{ij}(\mathbf{r}_{nm}) + 2\kappa \delta^{ij}] \nabla_{r_n^i} \nabla_{r_m^j}$$

is self-adjoint

$$\frac{d\langle f \rangle(\underline{\tilde{\boldsymbol{r}}}, t)}{dt} = \int \tilde{\mathcal{M}}_N f(\underline{\tilde{\boldsymbol{R}}}') \, \tilde{\mathcal{P}}_N(\underline{\tilde{\boldsymbol{r}}}; \underline{\tilde{\boldsymbol{R}}}'; t)) d\underline{\tilde{\boldsymbol{R}}}'$$

Zero modes

$$\tilde{\mathcal{M}}_N Z_N(\underline{\tilde{\boldsymbol{R}}}) = 0 \Longrightarrow \frac{d\langle Z_N \rangle(\underline{\tilde{\boldsymbol{r}}}, t)}{dt} = 0$$

$\xi = 0 \quad \tilde{\mathcal{M}}_N \to$ **multidimensional Laplacian** $\quad Z_N \to$ **Harmonic Polynomials**

e.g. $R_{12}^2 - R_{3,4}^2$

$N = 4 \quad 2(d+2)R_{12}^2 R_{34}^2 - d(R_{12}^4 + R_{34}^4)$

Zero modes are non-typical functions

Be $\underline{\boldsymbol{R}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$ **and** $\underline{\tilde{\boldsymbol{R}}}$ **the restriction to variables only depending on relative separations of N particles**

N particles are characterized by the size R **and the shape** $\dfrac{\underline{\tilde{\boldsymbol{R}}}}{R}$

Kraichnan Model

$$\partial_t \tilde{\mathcal{P}}_N = \tilde{\mathcal{M}}_N \tilde{\mathcal{P}}_N$$

$$\tilde{\mathcal{M}}_N = -\sum_{n<m} [\underbrace{d^{ij}(\mathbf{r}_{nm})}_{\sim r_{nm}^{\xi}} + 2\kappa\delta^{ij}]\nabla_{r_n^i}\nabla_{r_m^j}$$

$$[\tilde{\mathcal{M}}_N] = [L]^{\xi-2} \Longrightarrow [T] \sim [L]^{2-\xi}$$

$$\mathcal{P}_N(\underline{\tilde{\boldsymbol{R}}}, t) = \lim_{r\to 0} \mathcal{P}_N(\underline{\tilde{\mathbf{r}}}; \underline{\tilde{\boldsymbol{R}}}; t) \qquad 0 \le \xi < 2$$

$$\mathcal{P}_N(\underline{\tilde{\boldsymbol{R}}}, t) = \lambda^{(N-1)d} \mathcal{P}_N(\lambda\underline{\tilde{\boldsymbol{R}}}, \lambda^{2-\xi} t)$$

Be f a scaling function

$$f(\lambda\underline{\tilde{\boldsymbol{R}}}) = \lambda^{\sigma} f(\underline{\tilde{\boldsymbol{R}}})$$

$$\langle f \rangle = \int f(\underline{\tilde{\boldsymbol{R}}}) \mathcal{P}_N(\underline{\tilde{\boldsymbol{R}}}, t) d\underline{\tilde{\boldsymbol{R}}} = \lambda^{-\sigma} \int f(\lambda\underline{\tilde{\boldsymbol{R}}}) \lambda^{(N-1)d} \mathcal{P}_N(\lambda\underline{\tilde{\boldsymbol{R}}}, \lambda^{2-\xi} t) d\underline{\tilde{\boldsymbol{R}}}$$

$$= \lambda^{-\sigma} \int f(\lambda\underline{\tilde{\boldsymbol{R}}}) \mathcal{P}_N(\lambda\underline{\tilde{\boldsymbol{R}}}, \lambda^{2-\xi} t) d\underline{\lambda\tilde{\boldsymbol{R}}} = \lambda^{-\sigma} \int f(\underline{\tilde{\boldsymbol{R}}}) \mathcal{P}_N(\underline{\tilde{\boldsymbol{R}}}, \lambda^{2-\xi} t) d\underline{\tilde{\boldsymbol{R}}}$$

$$\lambda = t^{-1/(2-\xi)}$$

$$= t^{\frac{\sigma}{2-\xi}} \underbrace{\int f(\underline{\tilde{\boldsymbol{R}}}) \mathcal{P}_N(\underline{\tilde{\boldsymbol{R}}}, 1) d\underline{\tilde{\boldsymbol{R}}}}_{\text{depends only on the shape}}$$

A generic scaling function grows in time a là Richardson in order to be a zero mode the shape evolution should compensante for the size growth

Multi-particle separation

In the Kraichnan model zero modes exist also for $\xi \neq 0/2$

And are expected to exist also for generic velocity fields (i.e. non reversible, non Markovian and non Gaussian, though for such (general velocities) nobody knows the operator

In non-smooth flows the combination of explosive separation and zero modes has important consequences on the transport of substances

Summary and take home messages

- Single particle dispersion is typically diffusive unless long anomalous correlations are present
- two (N=2) or many particle (N>2) dispersion probe the spatial statistics of velocity field
- N=2 particle is qualitatively fitting Richardson picture though several difficulties make its experimental and numerical observation non trivial especially when interested to high order statistics
- **While in smooth velocity fields the Lagrangian flow is preserved in rough velocity fields one has explosive separation, the Lagrangian flow breaks and become spontaneously stochastic**
- N>2 particle dispersion leads to account not only for the evolution of the size but also of the shape which is in general non trivial
- There is a clean framework (The Kraichnan model) where both N=2 and N>2 particle dispersion can be studied
- **The study of the Kraichnan model highlights the existence of zero modes: function of relative separation of the particles statistically conserved by the Lagrangian dynamics**

Bibliography

Reviews

Sawford, Brian. "Turbulent relative dispersion." *Annual review of fluid mechanics* 33.1 (2001): 289-317.
Salazar, Juan PLC, and Lance R. Collins. "Two-particle dispersion in isotropic turbulent flows." *Annual review of fluid mechanics* 41.1 (2009): 405-432.
Falkovich, Gregory, K. Gawędzki, and Massimo Vergassola. "Particles and fields in fluid turbulence." *Reviews of modern Physics* 73.4 (2001): 913.

Theory/Modeling

Richardson, Lewis Fry. "Atmospheric diffusion shown on a distance-neighbour graph." *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 110.756 (1926): 709-737.
Batchelor, George Keith. "Diffusion in a field of homogeneous turbulence: II. The relative motion of particles." *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 48. No. 2. Cambridge University Press, 1952.
Kraichnan, Robert H. "Dispersion of particle pairs in homogeneous turbulence." *The Physics of Fluids* 9.10 (1966): 1937-1943.
Boffetta, Guido, et al. "Pair dispersion in synthetic fully developed turbulence." *Physical Review E* 60.6 (1999): 6734.
Sokolov, I. M. "Two-particle dispersion by correlated random velocity fields." *Physical Review E* 60.5 (1999): 5528.
Rast, Mark Peter, and Jean-François Pinton. "Pair dispersion in turbulence: the subdominant role of scaling." *Physical review letters* 107.21 (2011): 214501.
Thalabard, S, G Krstulovic, and J Bec. "Turbulent pair dispersion as a continuous-time random walk." *Journal of Fluid Mechanics* 755 (2014): R4.
Eyink, Gregory L., and Dmytro Bandak. "Renormalization group approach to spontaneous stochasticity." *Physical Review Research* 2, no. 4 (2020): 043161.
A. L Peixoto Considera, and S Thalabard. "Spontaneous stochasticity in the presence of intermittency." *Physical Review Letters* 131.6 (2023): 064001.

Experiments

Jullien, Marie-Caroline, Jérôme Paret, and Patrick Tabeling. "Richardson pair dispersion in two-dimensional turbulence." *Physical review letters* 82.14 (1999): 2872.
Ott, Søren, and Jakob Mann. "An experimental investigation of the relative diffusion of particle pairs in three-dimensional turbulent flow." *Journal of Fluid Mechanics* 422 (2000): 207-223.
Bourgoin, Mickaël, et al. "The role of pair dispersion in turbulent flow." *Science* 311.5762 (2006): 835-838.
Ouellette, Nicholas T., et al. "An experimental study of turbulent relative dispersion models." *New Journal of Physics* 8.6 (2006): 109.
Berg, Jacob, et al. "Backwards and forwards relative dispersion in turbulent flow: an experimental investigation." *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics* 74.1 (2006): 016304.
Shnapp, Ron, Stefano Brizzolara, Marius M. Neamtu-Halic, Alessandro Gambino, and Markus Holzner. "Universal alignment in turbulent pair dispersion." *Nature Communications* 14, no. 1 (2023): 4195.
Jucha, Jennifer, et al. "Time-reversal-symmetry breaking in turbulence." *Physical review letters* 113.5 (2014): 054501.

Bibliography

DNS-Richardson

Boffetta, Guido, and Igor M. Sokolov. "Relative dispersion in fully developed turbulence: the Richardson's law and intermittency corrections." *Physical review letters* 88.9 (2002): 094501.
Ishihara, T., & Kaneda, Y. (2002). Relative diffusion of a pair of fluid particles in the inertial subrange of turbulence. *Physics of Fluids*, *14*(11), L69-L72.
Boffetta, Guido, and Igor M. Sokolov. "Statistics of two-particle dispersion in two-dimensional turbulence." *Physics of fluids* 14.9 (2002): 3224-3232.
Yeung, P. K., & Borgas, M. S. (2004). Relative dispersion in isotropic turbulence. Part 1. Direct numerical simulations and Reynolds-number dependence. *Journal of Fluid Mechanics*, *503*, 93-124.
Biferale, L., Boffetta, G., Celani, A., Devenish, B. J., Lanotte, A., & Toschi, F. (2005). Lagrangian statistics of particle pairs in homogeneous isotropic turbulence. *Physics of Fluids*, *17*(11).
Bitane, R., Homann, H., & Bec, J. (2012). Time scales of turbulent relative dispersion. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, *86*(4), 045302.
Bitane, R., Homann, H., & Bec, J. (2013). Geometry and violent events in turbulent pair dispersion. *Journal of Turbulence*, *14*(2), 23-45.
Biferale, L., Lanotte, A. S., Scatamacchia, R., & Toschi, F. (2014). Intermittency in the relative separations of tracers and of heavy particles in turbulent flows. *Journal of fluid mechanics*, *757*, 550-572.
Buaria, D., Sawford, B. L., & Yeung, P. K. (2015). Characteristics of backward and forward two-particle relative dispersion in turbulence at different Reynolds numbers. *Physics of Fluids*, *27*(10).

Multi-Particle

Bernard, Denis, Krzysztof Gawedzki, and Antti Kupiainen. "Slow modes in passive advection." *Journal of Statistical Physics* 90, no. 3 (1998): 519-569.
Biferale, L., Boffetta, G., Celani, A., Devenish, B. J., Lanotte, A., & Toschi, F. (2005). Multiparticle dispersion in fully developed turbulence. *Physics of Fluids*, *17*(11).
Xu, H., Ouellette, N. T., & Bodenschatz, E. (2008). Evolution of geometric structures in intense turbulence. *New Journal of Physics*, *10*(1), 013012.
Pumir, Alain, Eberhard Bodenschatz, and Haitao Xu. "Tetrahedron deformation and alignment of perceived vorticity and strain in a turbulent flow." *Physics of Fluids* 25, no. 3 (2013).
Lüthi, B., Ott, S., Berg, J., & Mann, J. (2007). Lagrangian multi-particle statistics. *Journal of Turbulence*, (8), N45.
Polanco, J. I., Arun, S., & Naso, A. (2023). Multiparticle Lagrangian statistics in homogeneous rotating turbulence. *Physical Review Fluids*, *8*(3), 034602.
Hackl, J. F., Pui-Kuen Yeung, and B. L. Sawford. "Multi-particle and tetrad statistics in numerical simulations of turbulent relative dispersion." *Physics of Fluids* 23, no. 6 (2011).
Mazzitelli, I. M., Toschi, F., & Lanotte, A. S. (2014). An accurate and efficient Lagrangian sub-grid model. *Physics of Fluids*, *26*(9).
Mazzitelli, I. M., F. Fornarelli, A. S. Lanotte, and P. Oresta. "Pair and multi-particle dispersion in numerical simulations of convective boundary layer turbulence." *Physics of Fluids* 26, no. 5 (2014).