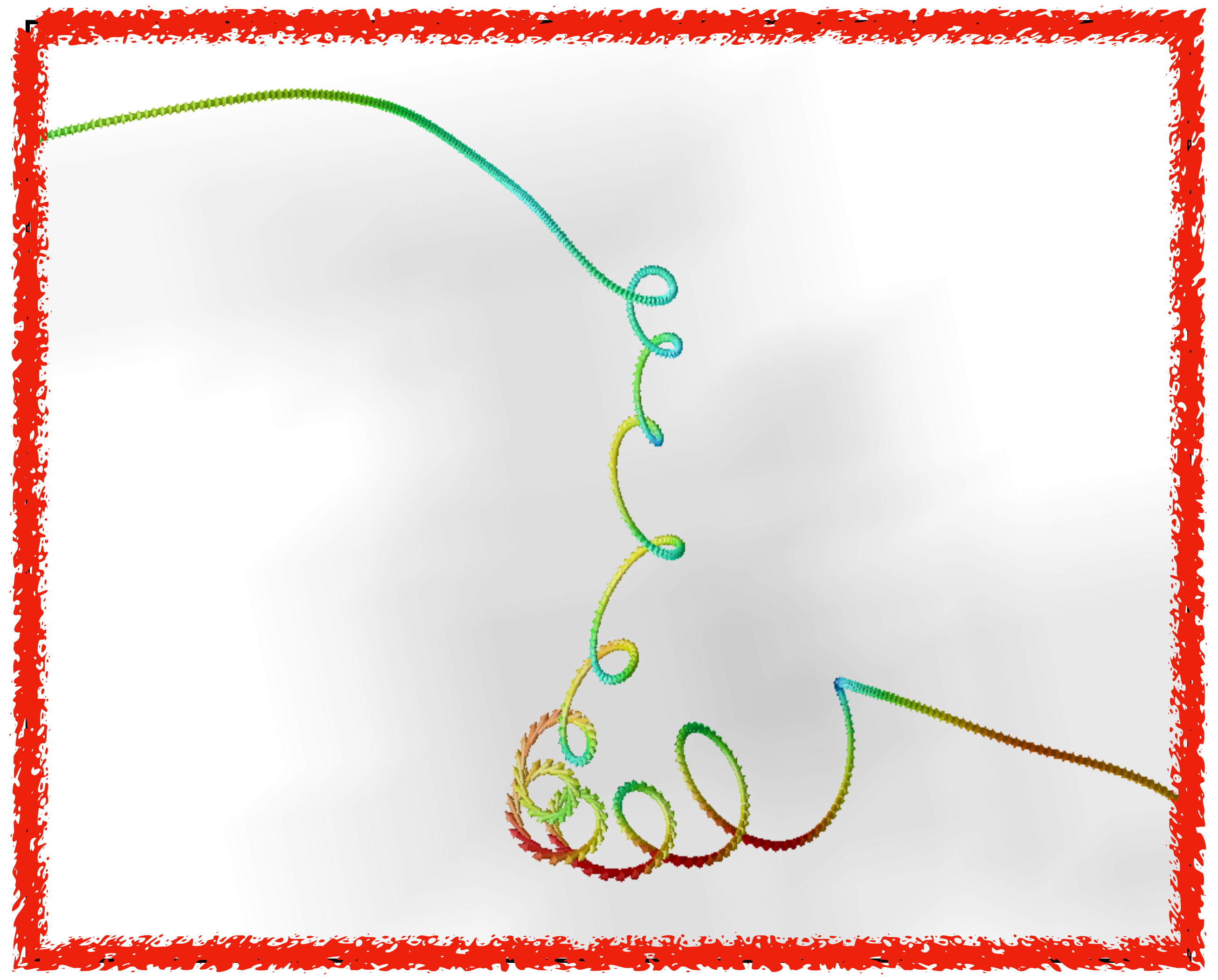
Lagrangian Turbulence: from tracers to intermittency and transport

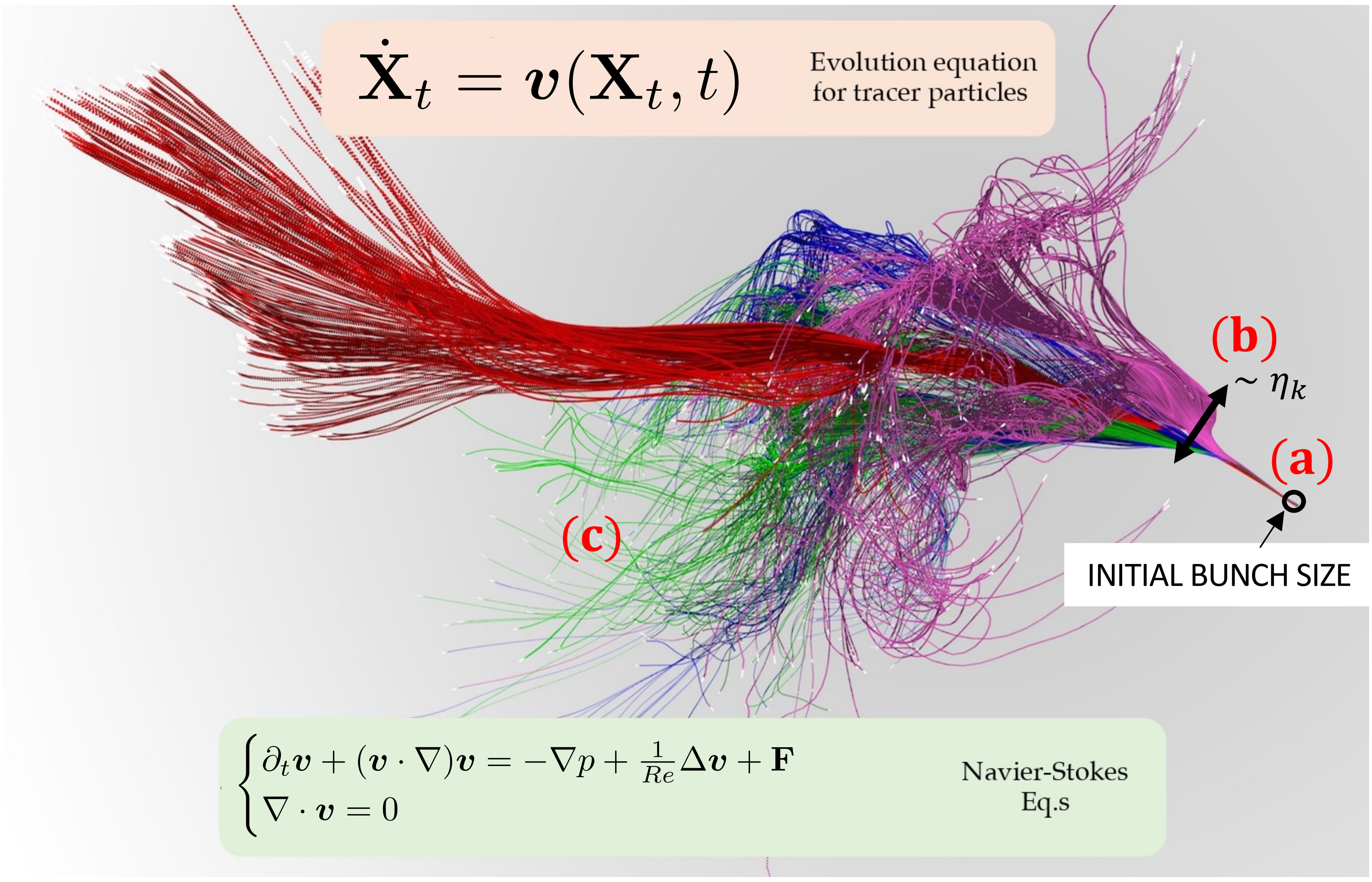
Massimo Cencini

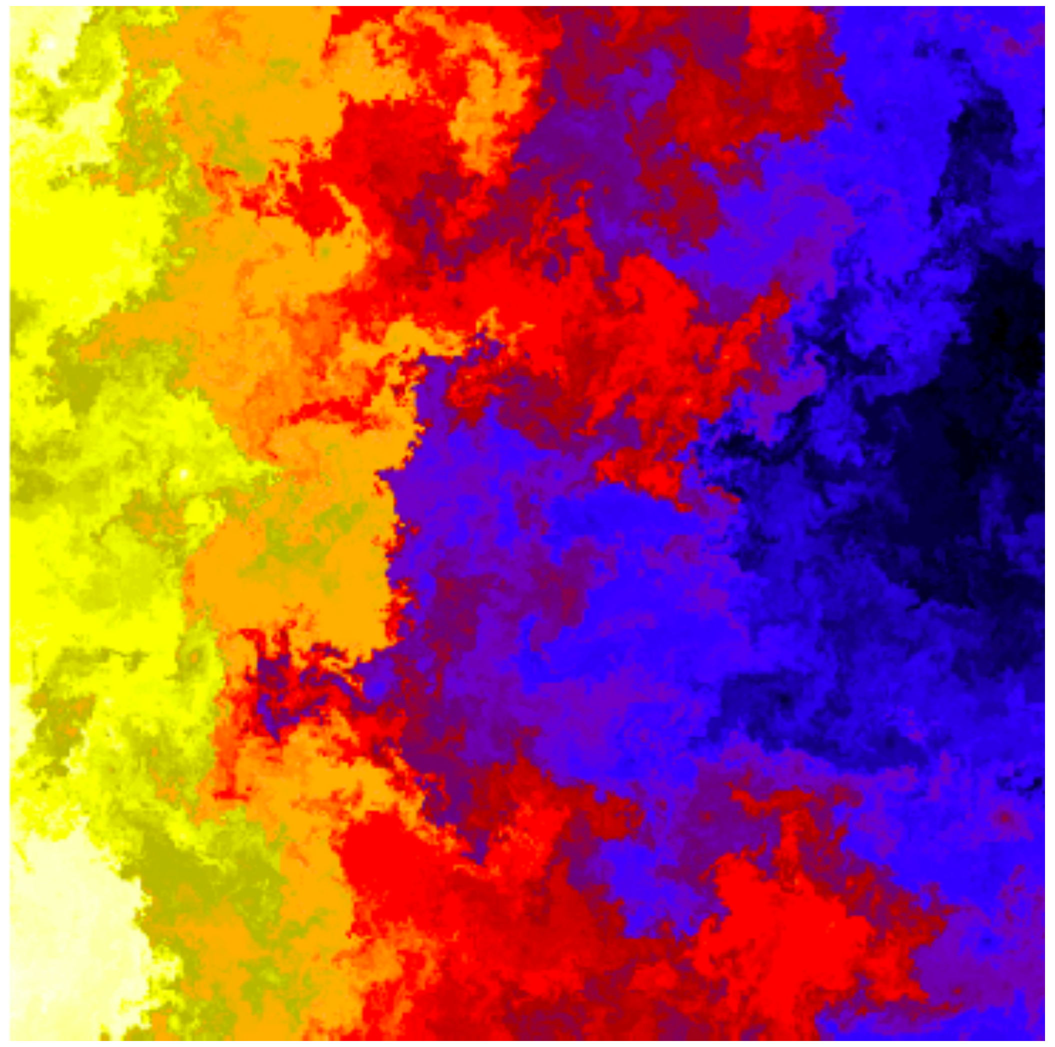
Istituto dei Sistemi Complessi CNR Rome, Italy

INFN “Tor Vergata”, Rome, Italy

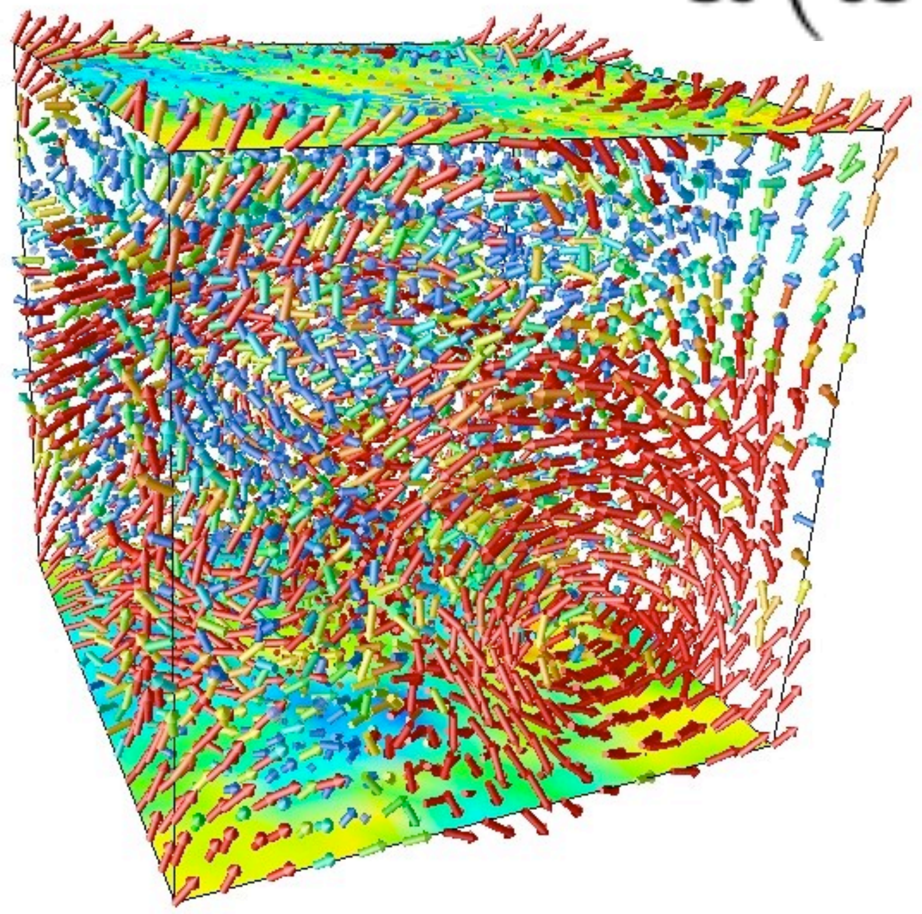
massimo.cencini@cnr.it





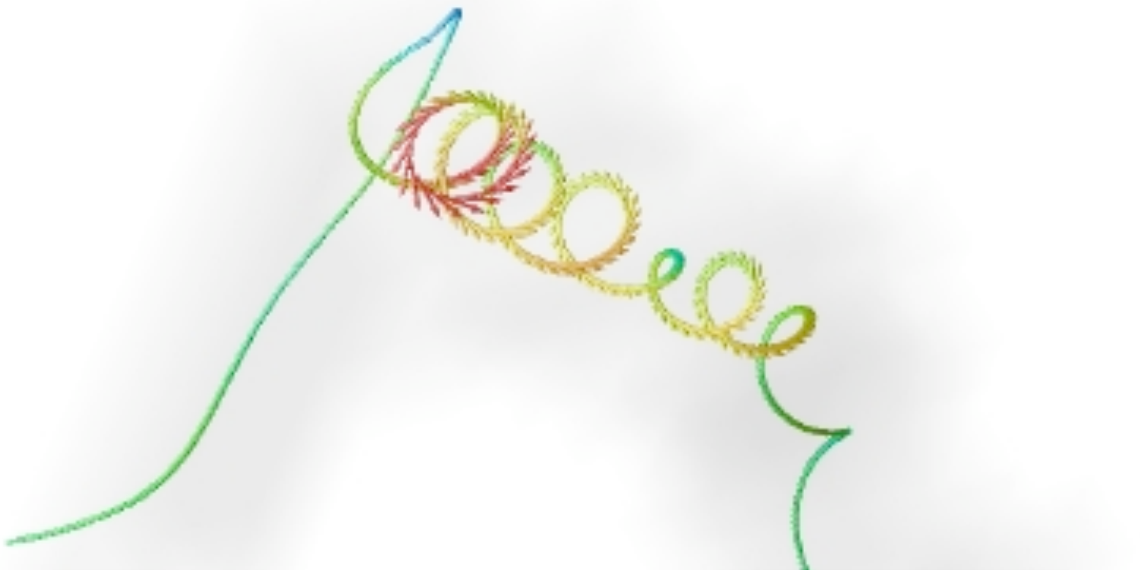


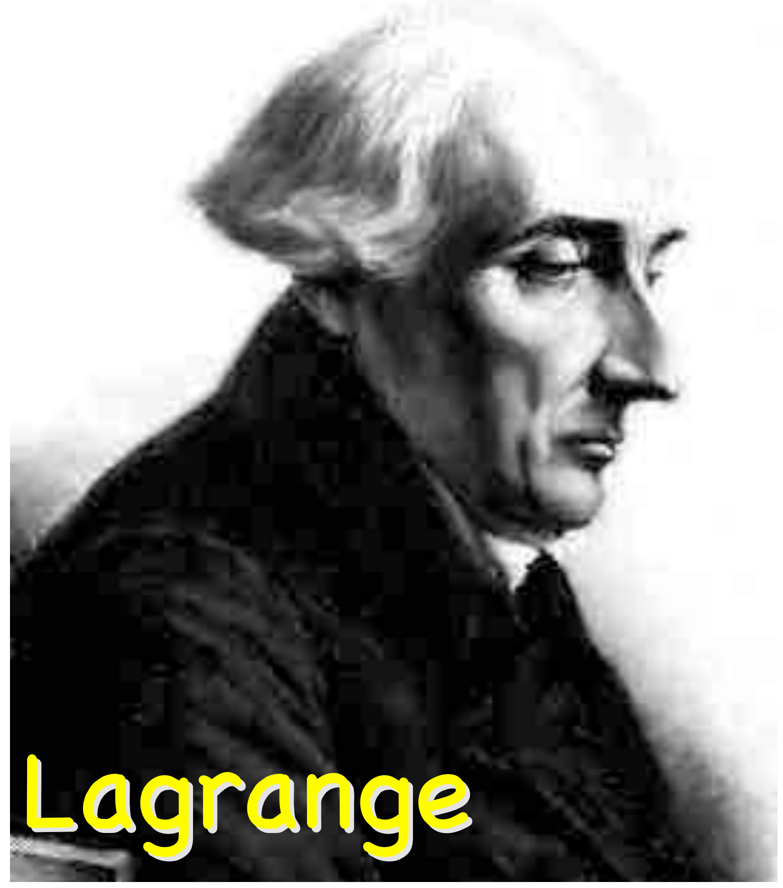
Two points of view



$$\boldsymbol{u}(\boldsymbol{x}, t)$$

$$\boldsymbol{X}(t) \quad \boldsymbol{u}(\boldsymbol{X}(t), t)$$





Evolution of the velocity fields

Evolution of fluid parcels (tracers)

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t)$$

While the focus will be on the Lagrangian point of view I will start recalling some basic phenomenology of turbulent velocity fields from an Eulerian point of view

Outline

- Recap: Basic facts on 3D Navier-Stokes (NS) turbulence, phenomenology of the energy cascade; Intermittency, anomalous scaling and the MultiFractal model (MF)

- An intermezzo: shell models, a gym for both Eulerian and Lagrangian turbulence

- Lagrangian turbulence: intermittency and MF model for velocity and acceleration statistics

- Lagrangian irreversibility (at the very last if there is time)

3D NS equation: very basic

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

in Fourier space

$$\partial_t \hat{\boldsymbol{u}}_{\boldsymbol{k}} = -i\boldsymbol{k}\mathbb{P}(\boldsymbol{k}) \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \hat{\boldsymbol{u}}_{\boldsymbol{p}} \hat{\boldsymbol{u}}_{\boldsymbol{q}} - \nu k^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \hat{\boldsymbol{F}}_{\boldsymbol{k}} \qquad \hat{\boldsymbol{u}}_{\boldsymbol{k}} \cdot \boldsymbol{k} = 0$$

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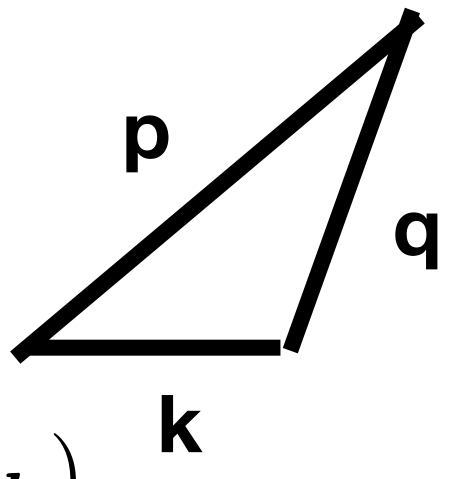
$$\boxed{\nu, F \to 0}$$

conserved

Energy $E = \frac{1}{2}\int d\boldsymbol{x} u^2 = \frac{1}{2}\sum_{\boldsymbol{k}} |\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2$

Helicity $H = \frac{1}{2}\int d\boldsymbol{x}\, \boldsymbol{u}\cdot\boldsymbol{\omega} = \frac{1}{2}\sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}\cdot\hat{\boldsymbol{\omega}}_{-\boldsymbol{k}}$

Conservations holds triad by triad



$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$$

phase space preserved $\dfrac{\partial(\partial_t \hat{\boldsymbol{u}}_{\boldsymbol{k}})}{\partial \hat{\boldsymbol{u}}_{\boldsymbol{k}}} = 0$

Equilibrium physics (if |k|<K <\infty)

3D NS equation: very basic

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

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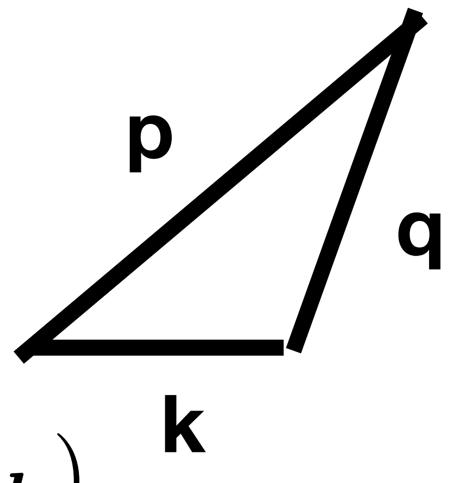

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Equilibrium physics (if |k|<K <\infty)

$$\boxed{\nu, F \neq 0}$$

$$\frac{dE}{dt} = \langle \boldsymbol{u}\cdot\boldsymbol{F}\rangle - \nu\langle|\boldsymbol{\nabla}\boldsymbol{u}|^2\rangle = \epsilon_{in} - \epsilon_{out} \approx 0$$

$$\epsilon = \epsilon_{in} = \epsilon_{out}$$

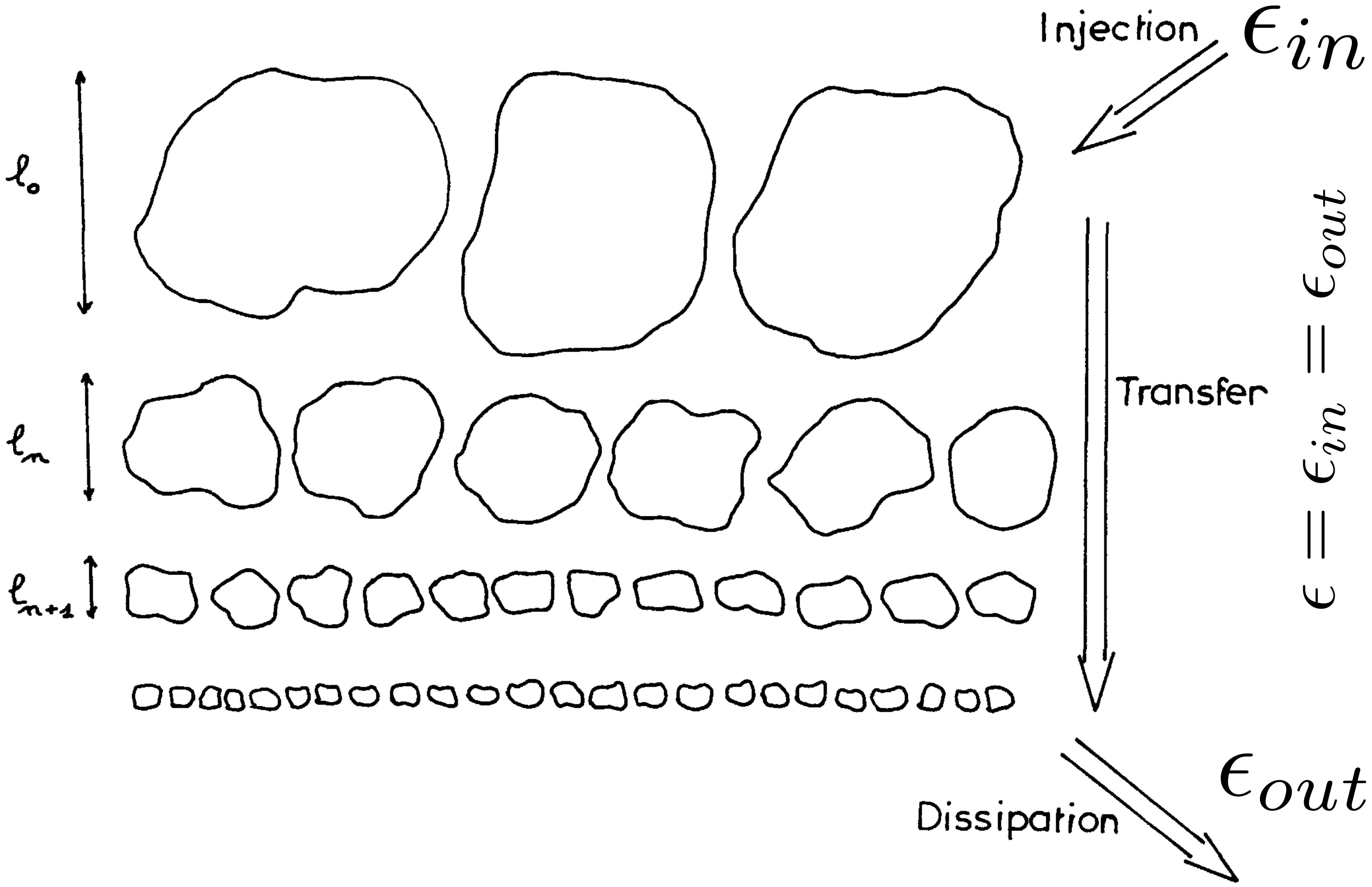
$$\lim_{\nu\to 0} \nu\langle|\boldsymbol{\nabla} u|^2\rangle = \epsilon > 0$$ **dissipative anomaly**

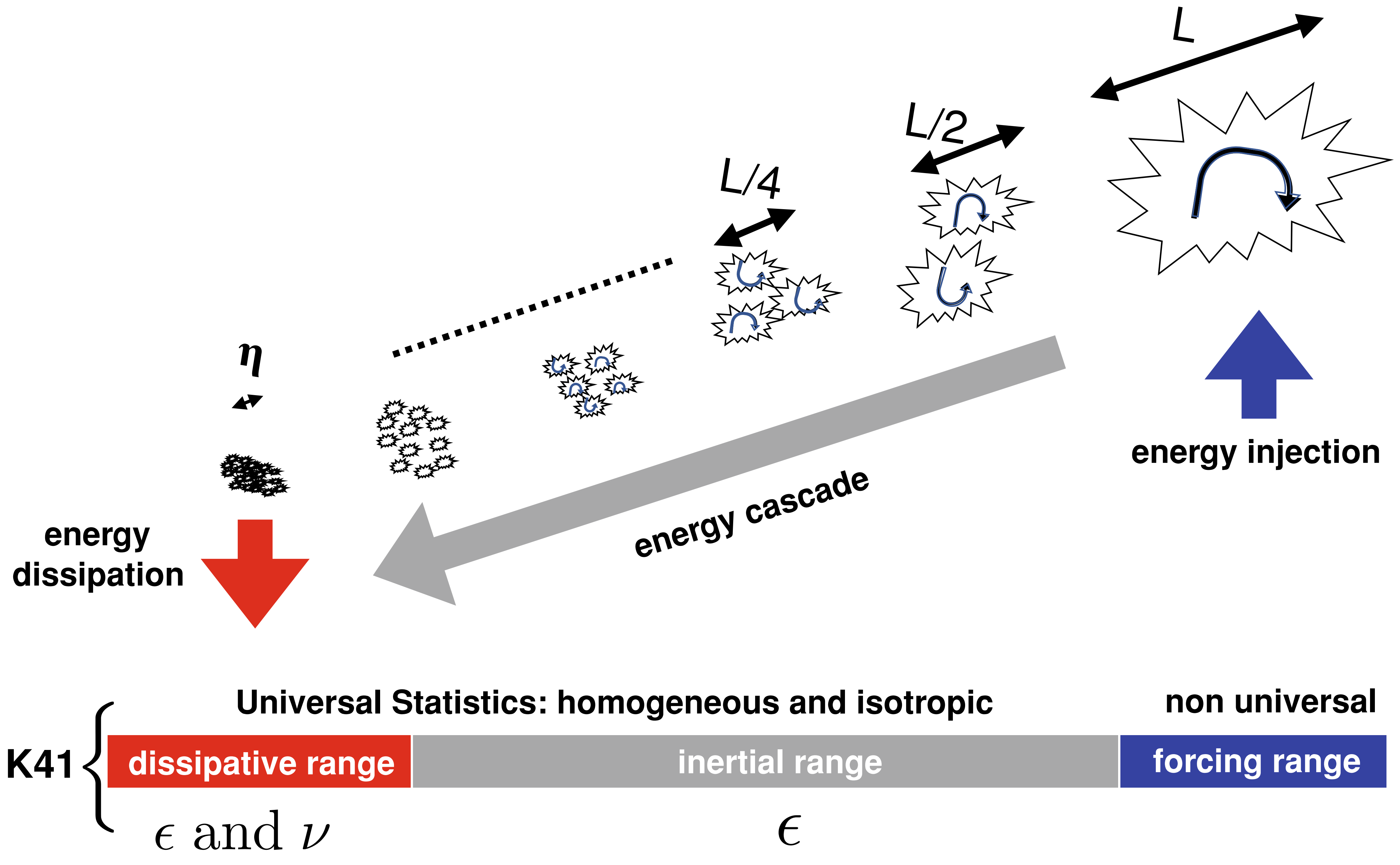
Non-Equilibrium Statistically Steady State

with flow (cascade) of energy

NS turbulence: (3D) energy cascade

$$\frac{dE}{dt} = \langle \boldsymbol{u} \cdot \boldsymbol{F} \rangle - \nu \langle |\boldsymbol{\nabla} \boldsymbol{u}|^2 \rangle = \epsilon_{in} - \epsilon_{out} \approx 0 \qquad Re = \frac{UL}{\nu} \approx \frac{\textit{inertial terms}}{\textit{dissipative terms}} \longrightarrow \infty$$





Driven dissipative strongly nonlinear and non-equilibrium dynamical system with many characteristic scales and times

Kolmogorov 1941

Velocity increments

$$\delta_r u = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)]\cdot\hat{\boldsymbol{r}}$$

$$\langle(\delta_r u)^2\rangle \Longleftrightarrow E(k)$$

Far from dissipative and forcing scale the statistics is universal and only determined by the scale and ϵ

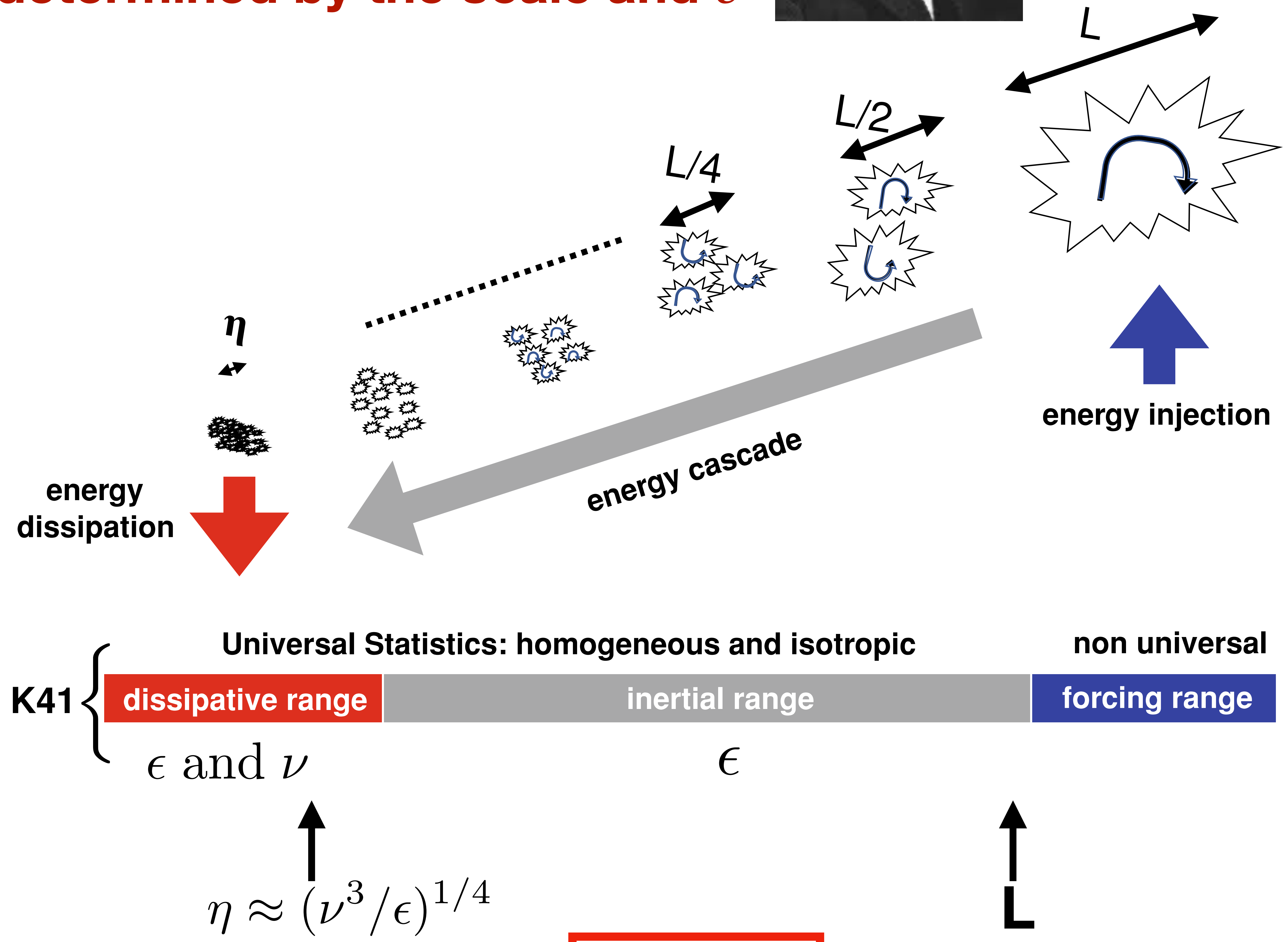
Universality + Dimensional analisys (K41)

$$[\epsilon] = [E]/[T] = [V^3][L] \qquad [E(k)] = [E]/[k] = [V^2]/[k]$$

$$\delta_r u \sim (\epsilon r)^{1/3} \qquad E(k) \sim \epsilon^{2/3}k^{-5/3}$$

$$\langle(\delta_r u)^p\rangle \sim (\epsilon r)^{p/3}$$

$$\eta \ll r \ll L$$



$$\boxed{\frac{L}{\eta} \sim Re^{3/4}}$$

Consistent with the exact result (yet from K in 1941)

$$S_3(r) = \langle(\delta_\parallel u(r))^3\rangle = -\frac{4}{5}\epsilon r + 6\nu\partial_r S_2(r) + ..$$

Kolmogorov 1941

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SCALING $\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F}$

For $\nu \to 0$
neglecting forcing
it holds a general
scaling symmetry

$$t, \boldsymbol{x}, \boldsymbol{u} \to \lambda^{1-h} t, \lambda \boldsymbol{x}, \lambda^h \boldsymbol{u}$$

K41- corresponds to global scaling invariance with h=1/3

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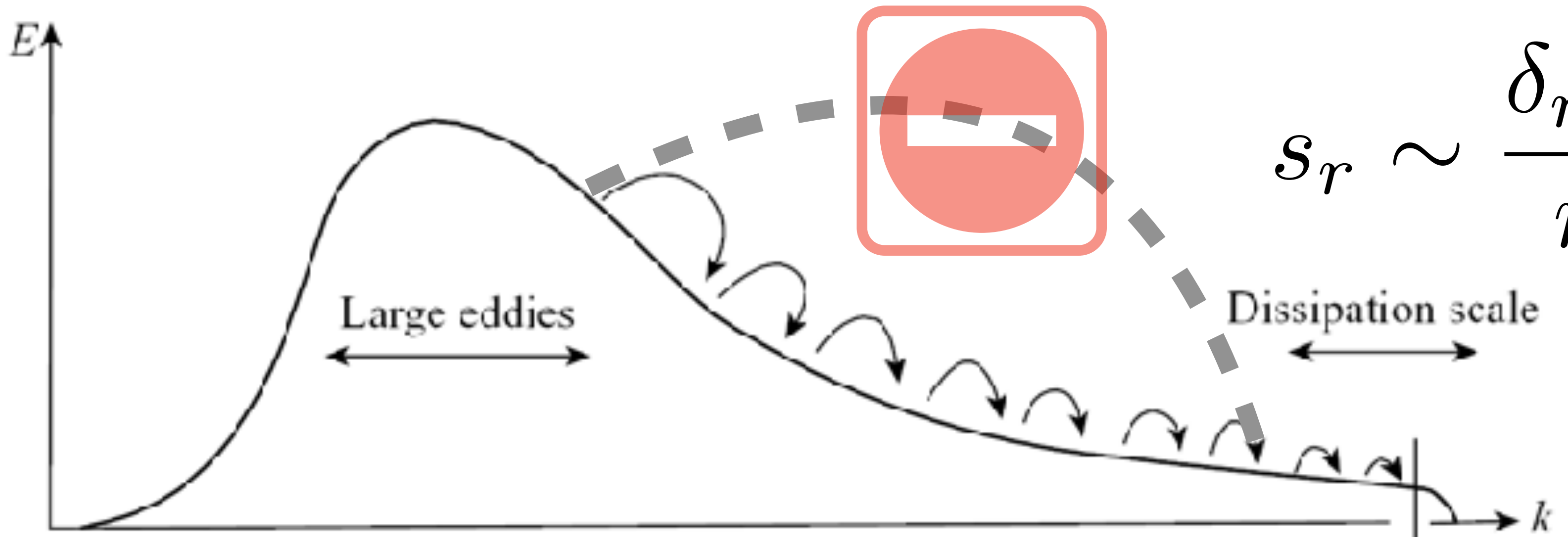
LOCALITY
Larger eddies->sweeping
Smaller eddies-> incoherent

eddies of size r are distorted (energy transferred) by eddies of comparable scale



$$s_r \sim \frac{\delta_r u}{r} \sim \epsilon^{1/3} r^{-2/3} \sim 1/\tau_r$$

Only triads with sides of comparable sides contribute to energy transfer

K41 against data

$$\langle(\delta_r u)^n\rangle \sim r^{\zeta_n}$$

$$P(\delta_r u/[\delta_r u]_{rms})$$

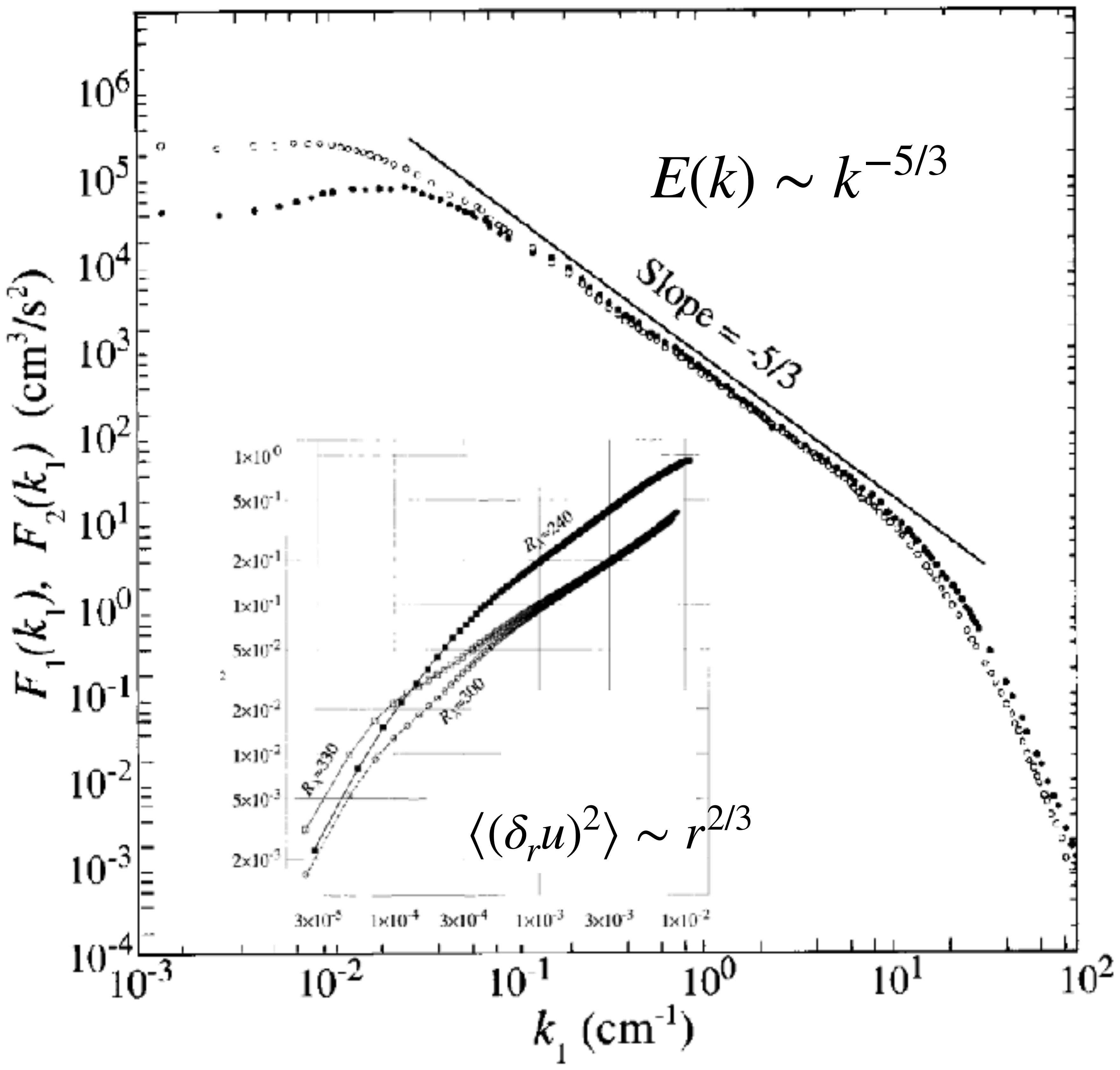
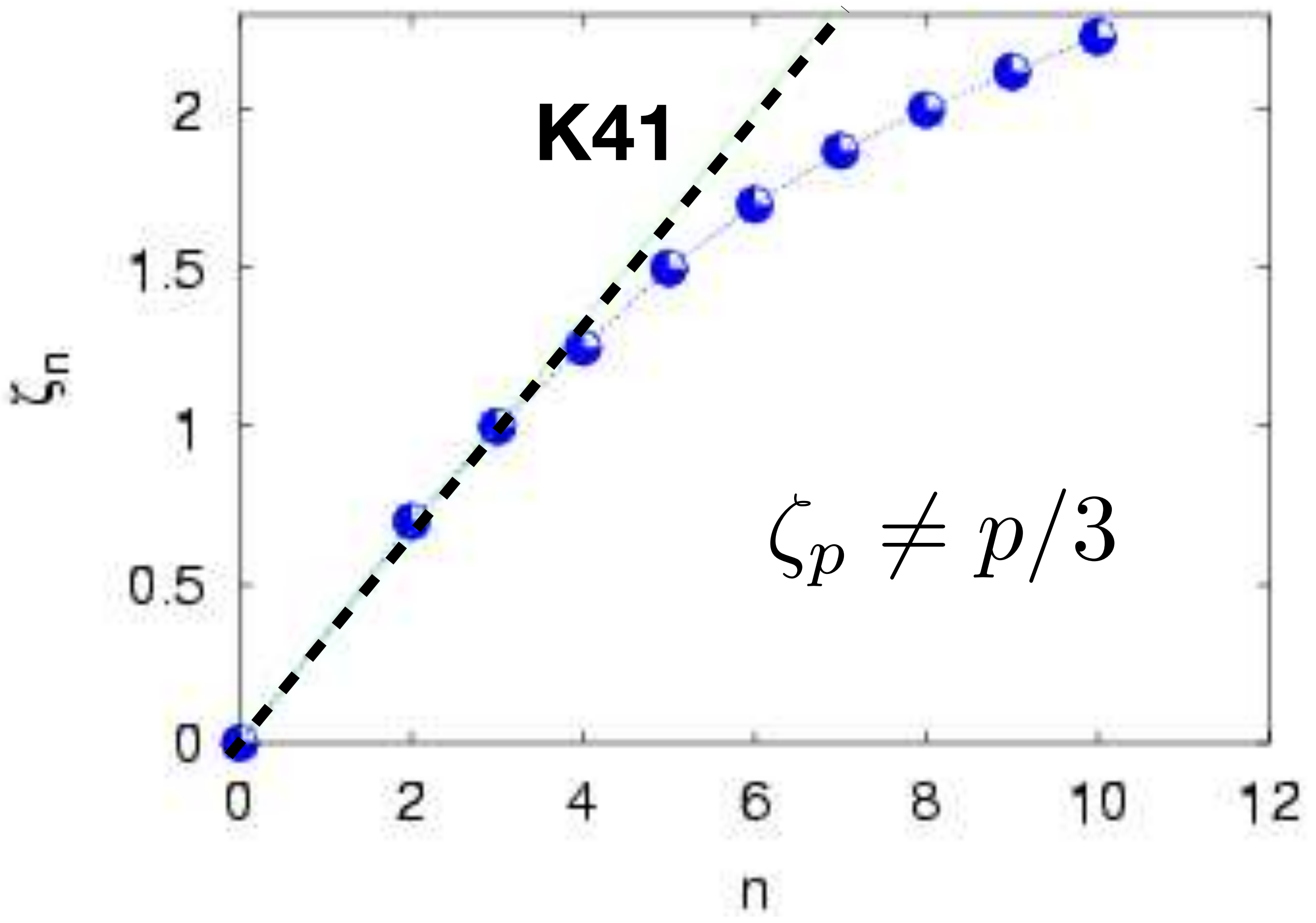
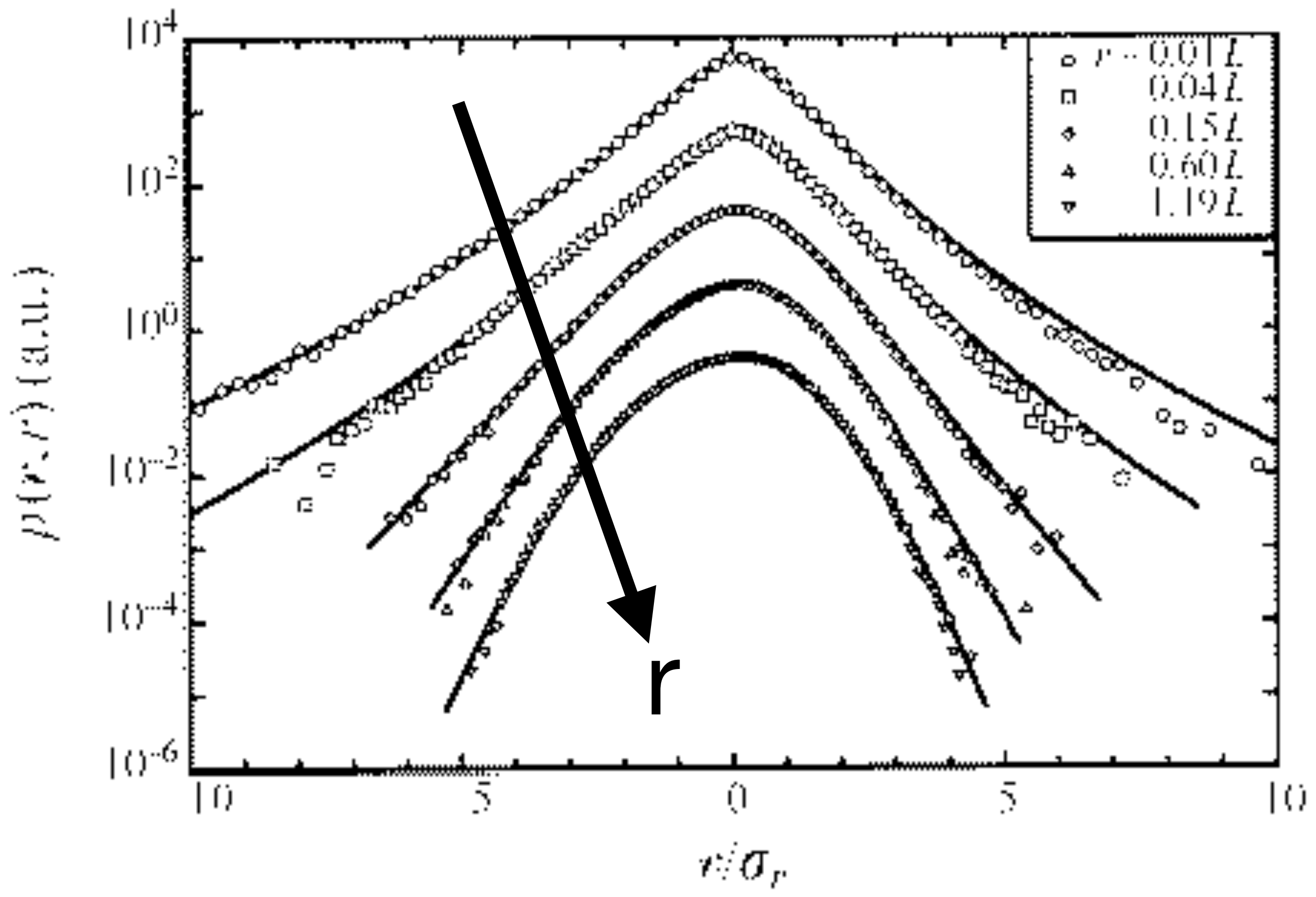


Fig. 5.7. log–log plot of the energy spectra of the streamwise component (white circles) and lateral component (black circles) of the velocity fluctuations in the time domain in a jet with $R_\lambda = 626$ (Champagne 1978).

Energy spectrum close to K41 prediction



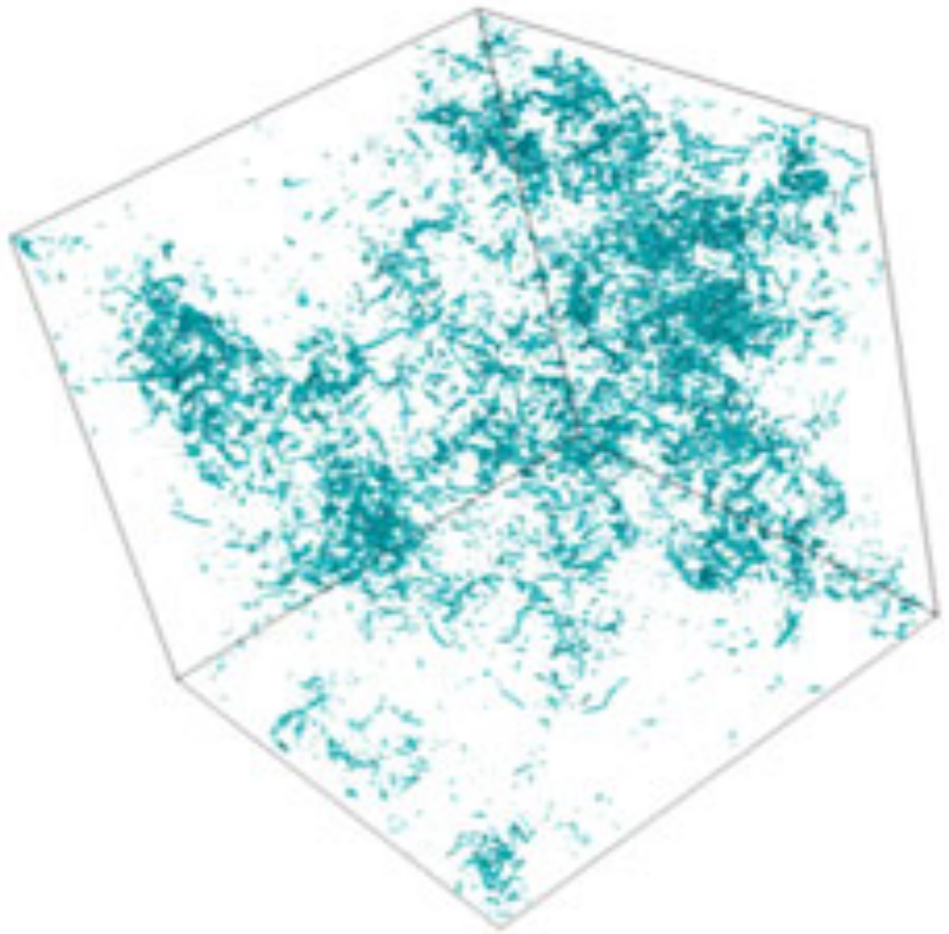


Higher moments deviate from K41—->intermittency

$$\langle(\delta_r u)^p\rangle \sim (\epsilon r)^{\zeta_p}$$

$$\zeta_p^{K41} = p/3 \qquad \zeta_p^{EXP+SIM} \neq p/3$$

This discrepancy is the hallmark of intermittency quantified by anomalous exponents



local energy dissipation highly inhomogeneous

Multifractal model

1983

Parisi **Frisch**

G. Parisi & U. Frisch. in *Turbulence and predictability in geophysical fluid dynamics and climate dynamics* (1985): 84-88.
*Proceedings of the Internutionul School of Physics *E.Femiu, Varenna, Italy*

$$\eta \ll r \ll L \qquad S_q(r) = \langle (\delta u(r))^q \rangle \sim \left(\frac{r}{L}\right)^{\zeta_q} \qquad \zeta_q \neq \frac{q}{3}$$

$$\delta u(r) \sim U \left(\frac{r}{L}\right)^h \qquad h(\boldsymbol{x}) = h \quad \boldsymbol{x} \in S_h \quad \text{fractal set with} \quad D(h) \qquad Prob_h(r) \sim \left(\frac{r}{L}\right)^{3-D(h)}$$

r

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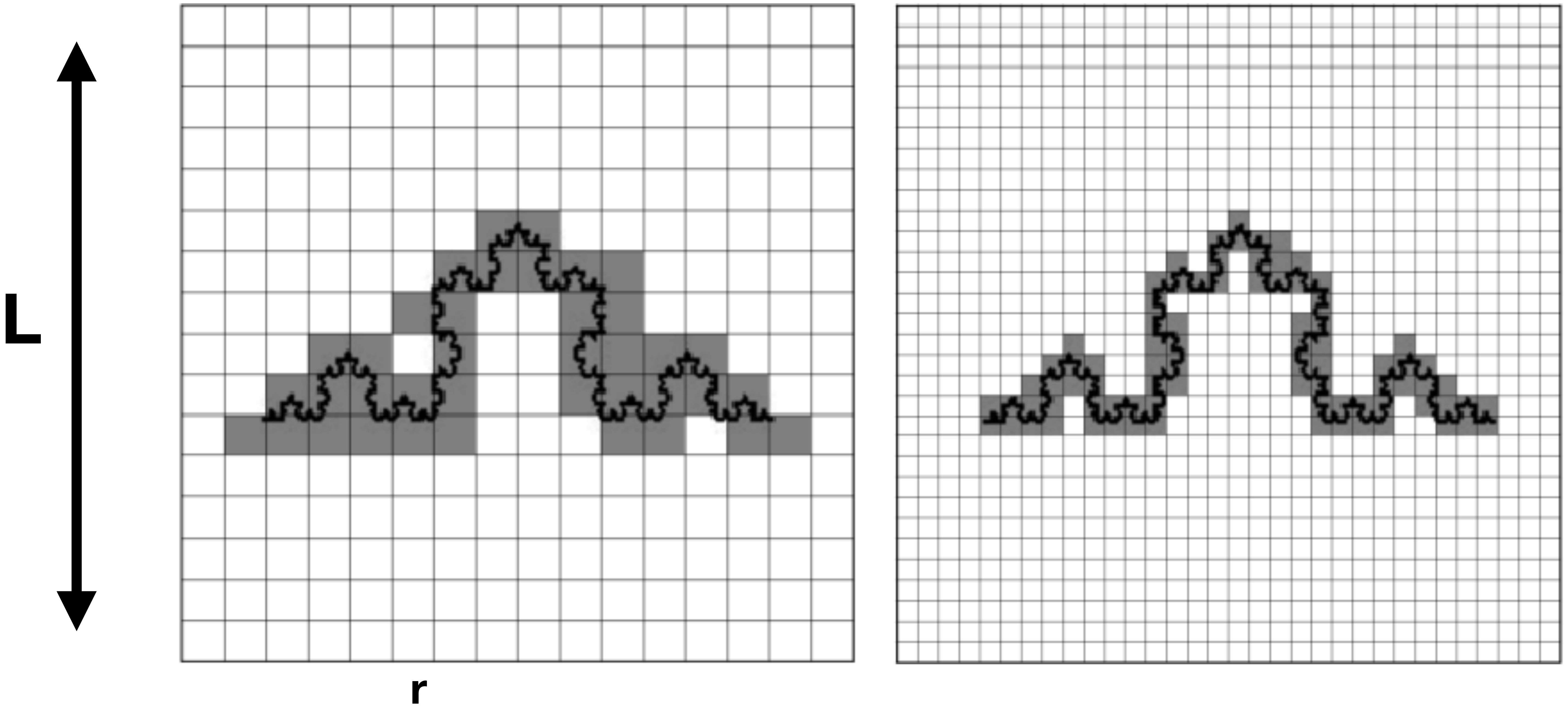

$$N_h(r) \sim (L/r)^{D(h)}$$

$$N_{tot}(r) \sim (L/r)^d$$

$$Prob_h(r) = \frac{N_h(r)}{N_{tot}(r)}$$

Multifractal model

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$$\langle (\delta_r u)^q \rangle = S_q(r) \sim U^q \int dh \left(\frac{r}{L}\right)^{hq+3-D(h)} \underset{\frac{r}{L} \to 0}{\sim} U^q \left(\frac{r}{L}\right)^{\zeta_q}$$

$$\zeta_q = \inf_h \{hq + 3 - D(h)\}$$

Laplace's method

K41 global scale invariance —> MF local scale invariance

K41 —-> h=1/3 D[1/3]=3

$$t, \boldsymbol{x}, \boldsymbol{u} \to \lambda^{1-h} t, \lambda \boldsymbol{x}, \lambda^h \boldsymbol{u}$$

A technical remark

The essence of Laplace's method is to assume that $ph + 3 - D(h)$ **has a quadratic minimum around a given value**

$$ph + 3 - D(h) \approx \zeta_p + A(h - h_p^*)^2$$

$$\int_{-\infty}^{\infty} dh \left(\frac{r}{L}\right)^{\zeta_p + A(h-h_p^*)^2} = \left(\frac{r}{L}\right)^{\zeta_p} \boxed{\int_{-\infty}^{\infty} dh \exp\left[-A \ln\left(\frac{L}{r}\right)(h - h_p^*)^2\right]}$$

$$\propto [\ln(L/r)]^{-1/2}$$

why such logarithmic corrections are never seen?

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why such logarithmic corrections are never seen?

Refined arguments based on rigorous application of Large Deviation Theory show that the probability to pick an h value should be actually written as

$$Prob_h(r) \sim \left[\ln\left(\frac{L}{r}\right)\right]^{1/2} \left(\frac{r}{L}\right)^{3-D(h)}$$

In the following we will ignore this technical point

FRISCH U, AFONSO MM, MAZZINO A, YAKHOT V. "Does multifractal theory of turbulence have logarithms in the scaling relations?" JFM 542:97-103 (2005)

How to determine D(h)?

The dream would be to have it from a theory based on the NS equation: but we do not have it

What we can easily to is to use the known (measured) moments of the velocity increments and from those obtain D(h) or a fit of it

Use some intuition and physics knowledge of the problem to have a fitting function with no or very few fitting parameters

For instance 4/5 law says us that

$$S_3(r) = \langle (\delta_{\parallel} u(r))^3 \rangle = -\frac{4}{5}\epsilon r + 6\nu \partial_r S_2(r) + \ldots \quad \Rightarrow \quad \zeta_3 = \inf_h \{3h + 3 - D(h)\} = 1$$

In the next slide we sketch the so-called (generalized) She-Leveque model

Z-S. She, E. Leveque. "Universal scaling laws in fully developed turbulence." PRL 72, 336 (1994)

She-Leveque model

$$D(h) = \frac{3(h-h_0)}{\log(\beta)}\left[\log\left(\frac{3(h_0-h)}{d_0\log(\beta)}\right)-1\right]+3-d_0$$

h_0 **is the exponent of the most singular structure**

d_0 **is the codimension of the set of the most singular structure**

$\beta < 1$ **a positive parameter tuning the statistics of the fluctuations (level of intermittency)**

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$$\zeta_p = \inf_h\{ph + 3 - D(h)\}$$

$$\Downarrow$$

$$D'(h_p^*) = p$$

$$\Leftarrow$$

$$h_p^* = h_0 - \frac{d_0\beta^{p/3}\log\beta}{3}$$

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4/5 law fixes one of the parameters

$$\zeta_3 = 1 \Longrightarrow d_0 = \frac{1-3h_0}{1-\beta}$$

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$$\zeta_p = h_0 p + \frac{(1-3h_0)}{(1-\beta)}(1-\beta^{p/3})$$

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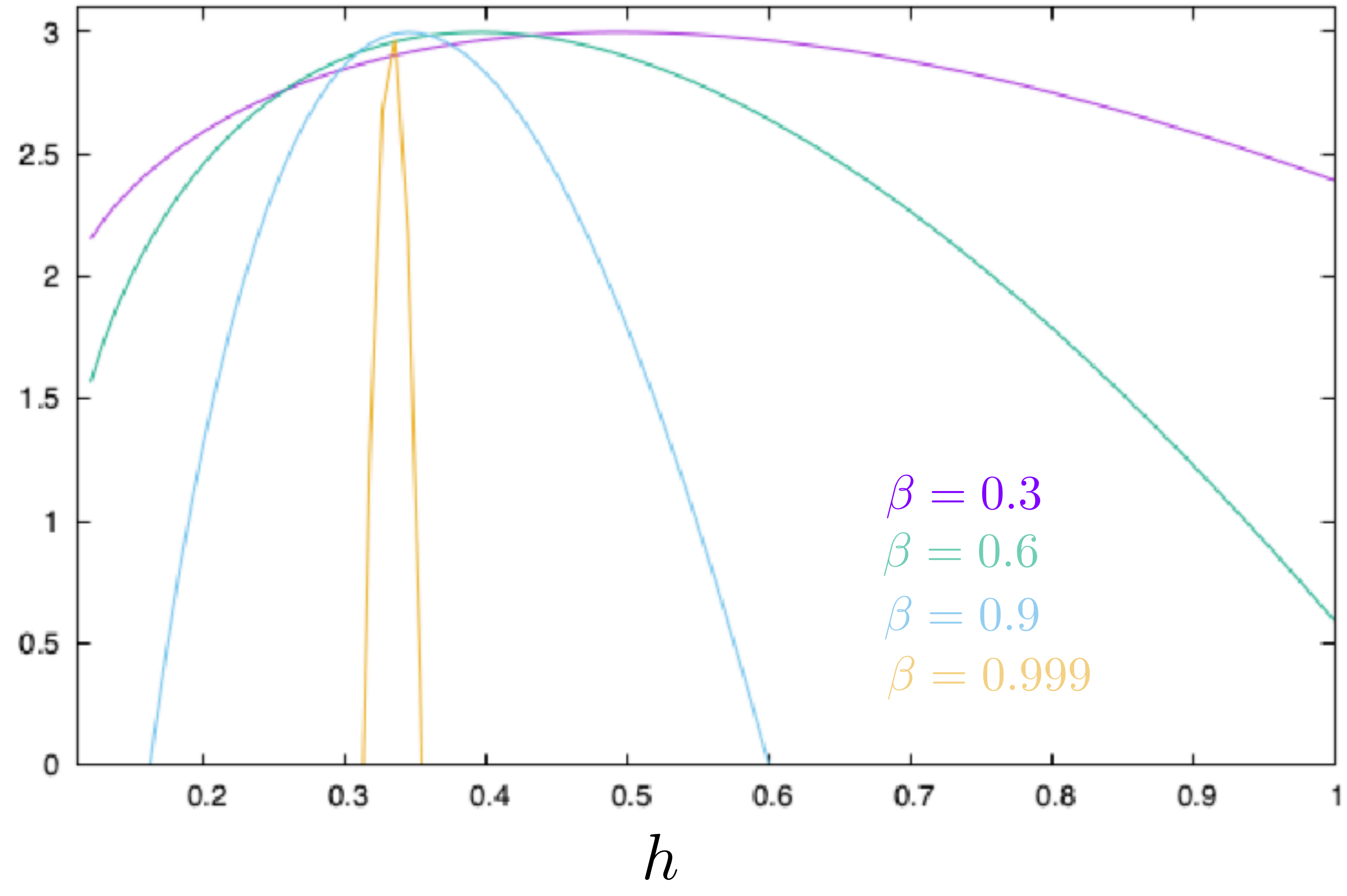
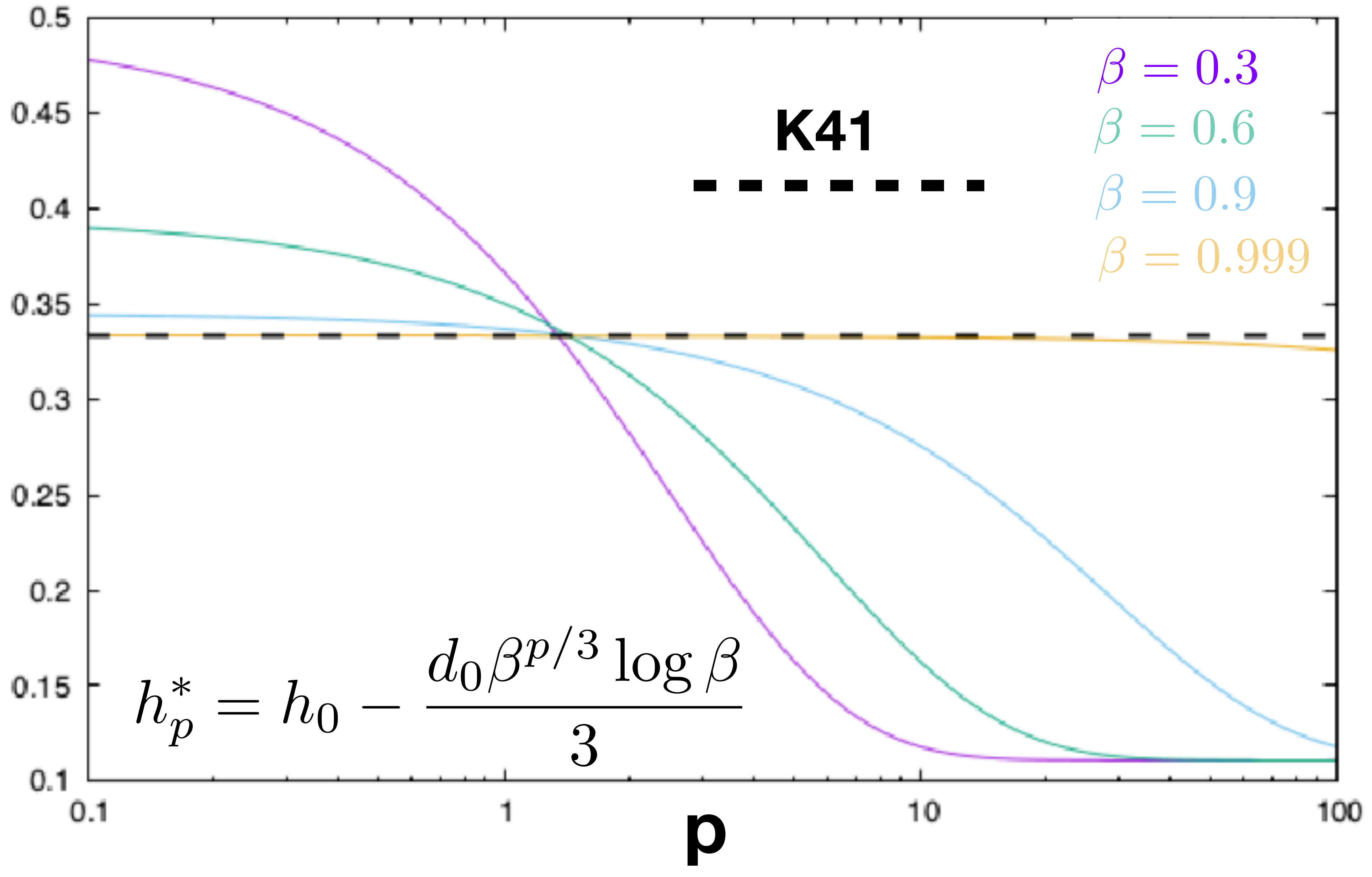
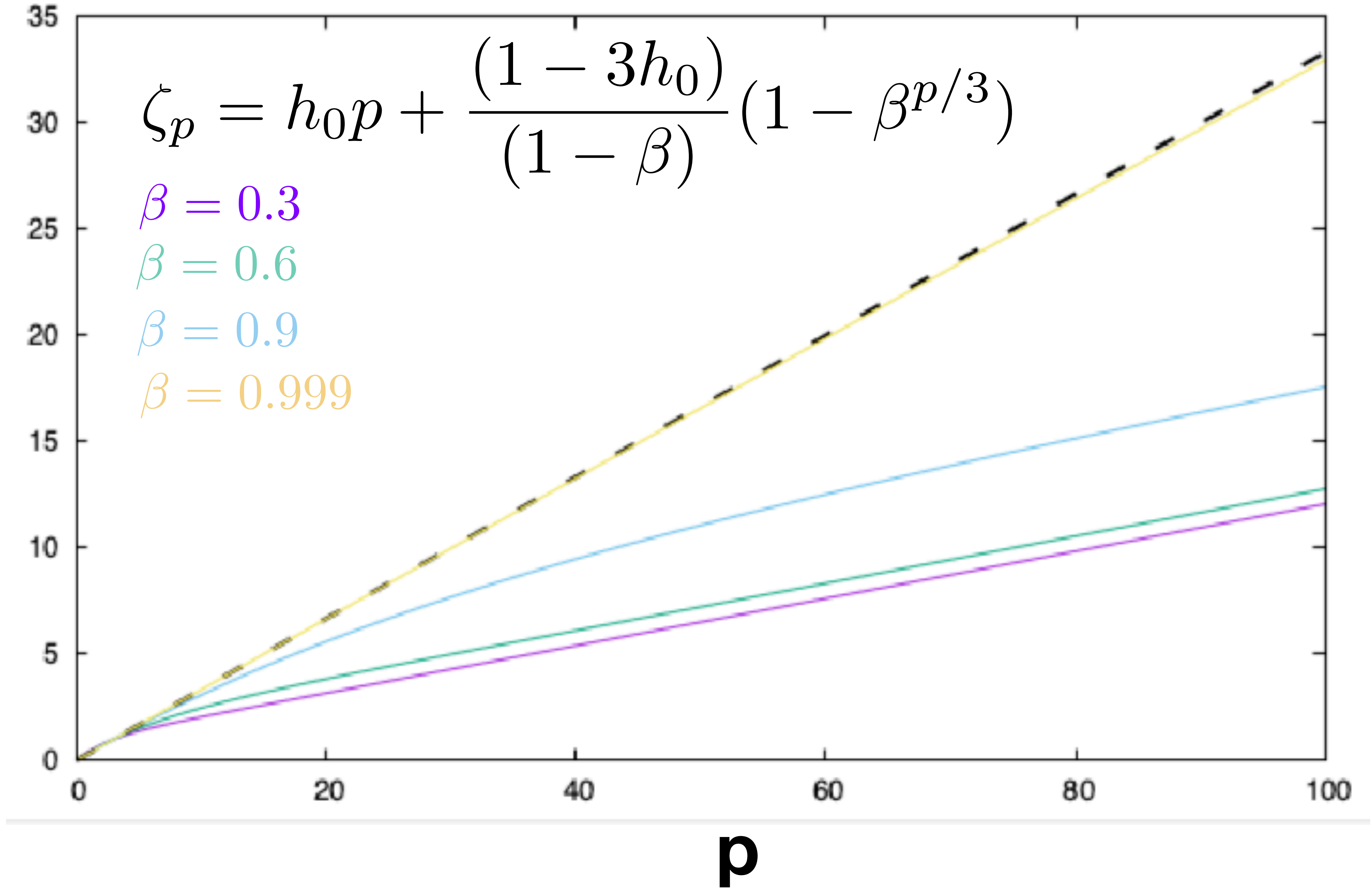
Visualizing MF

$$D(h) = \frac{3(h-h_0)}{\log(\beta)}\left[\log\left(\frac{3(h_0-h)}{d_0\log(\beta)}\right)-1\right]+3-d_0$$

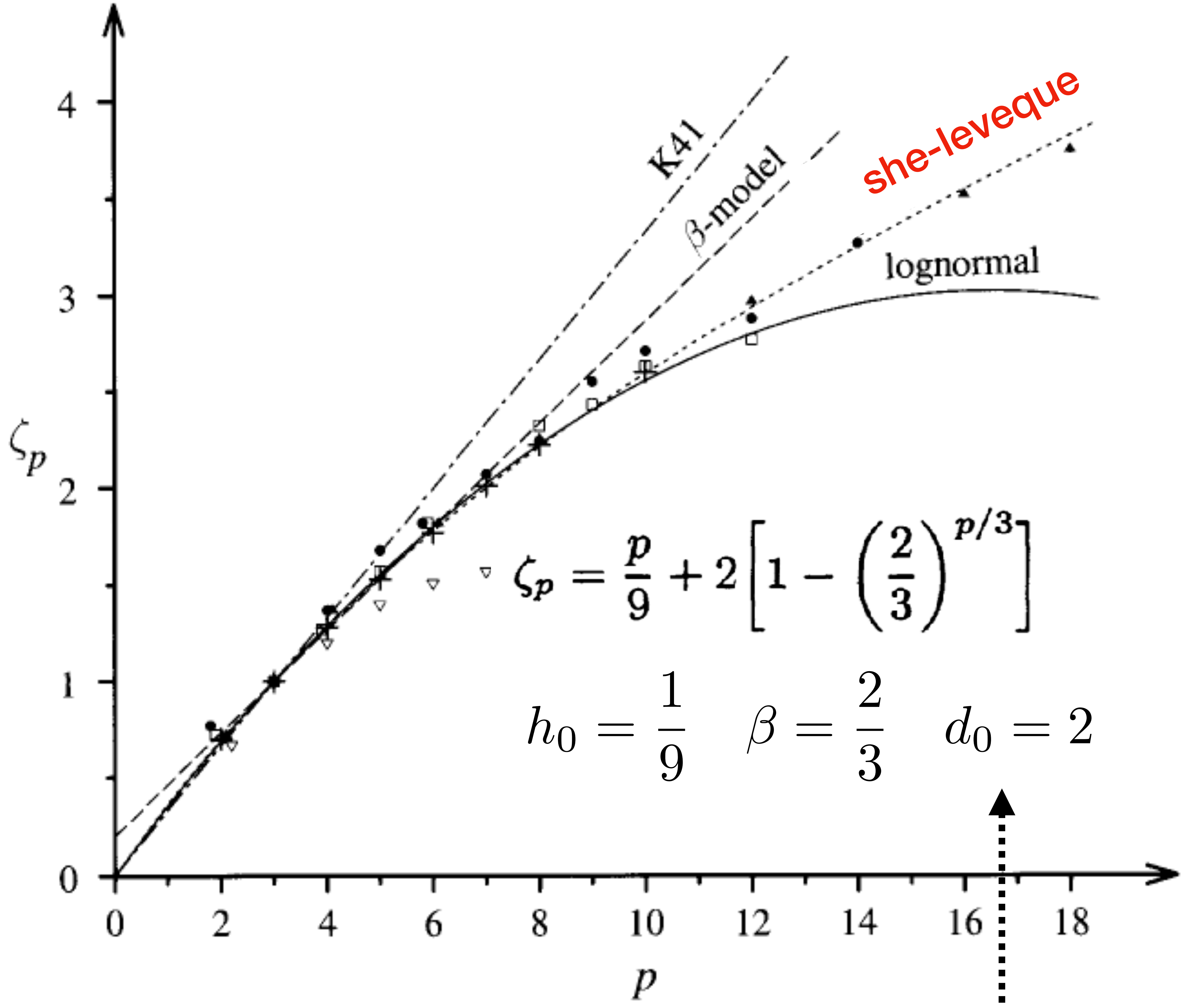


$$h_0 = \frac{1}{9}$$





Comparison with data



$$\zeta_p = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right]$$

$$h_0 = \frac{1}{9} \quad \beta = \frac{2}{3} \quad d_0 = 2$$

$$3 - d_0 = 1$$

dimensionality of vortex filaments

Z-S. She & E. Leveque. PRL 72, 336 (1994) B. Dubrulle PRL 73(7), 959 (1994) Z-S. She & E.C. Waymire. PRL 74 262 (1995)

A prediction of the MF model

The intermediate dissipative range

$$Re = \frac{UL}{\nu} \approx \frac{\color{red}{inertial\ terms}}{\color{blue}{dissipative\ terms}} \approx \frac{\delta_L u L}{\nu}$$

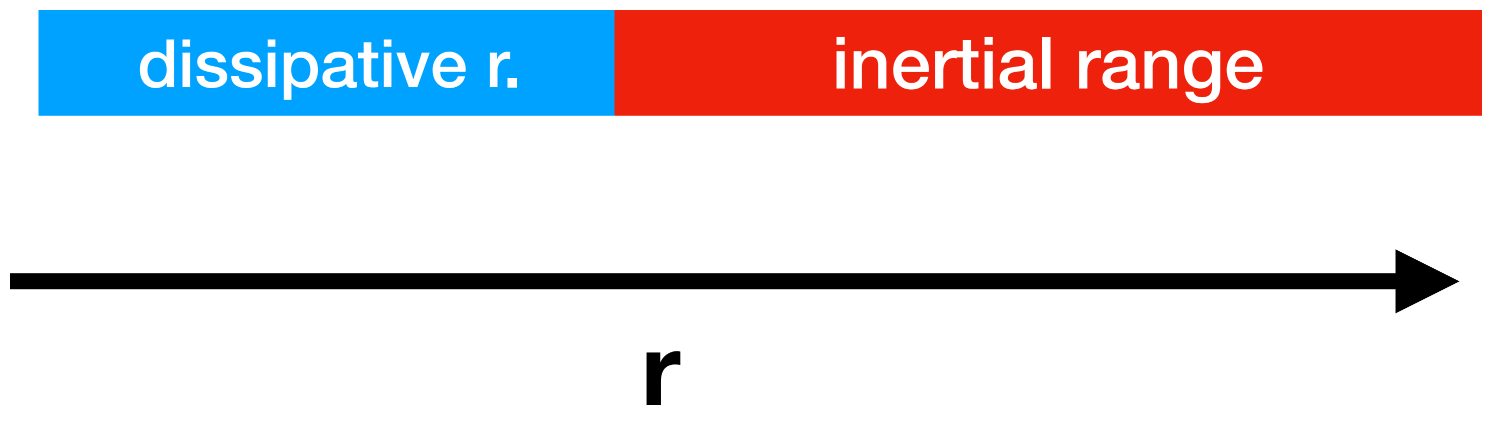
dissipative scale $$Re_\eta = \frac{\delta_\eta u\, \eta}{\nu} \approx 1$$

K41

$$\delta_r u \sim \delta_L u \left(\frac{r}{L}\right)^{1/3}$$

$$1 \approx \frac{\delta_\eta u \eta}{\nu} \approx \frac{\delta_L u L}{\nu}\left(\frac{\eta}{L}\right)^{1/3+1} = Re\left(\frac{\eta}{L}\right)^{4/3}$$

$$\left(\frac{\eta}{L}\right) \approx Re^{-3/4}$$



MF

U. Frisch & M. Vergassola EPL 14, 439 (1991)

A prediction of the MF model

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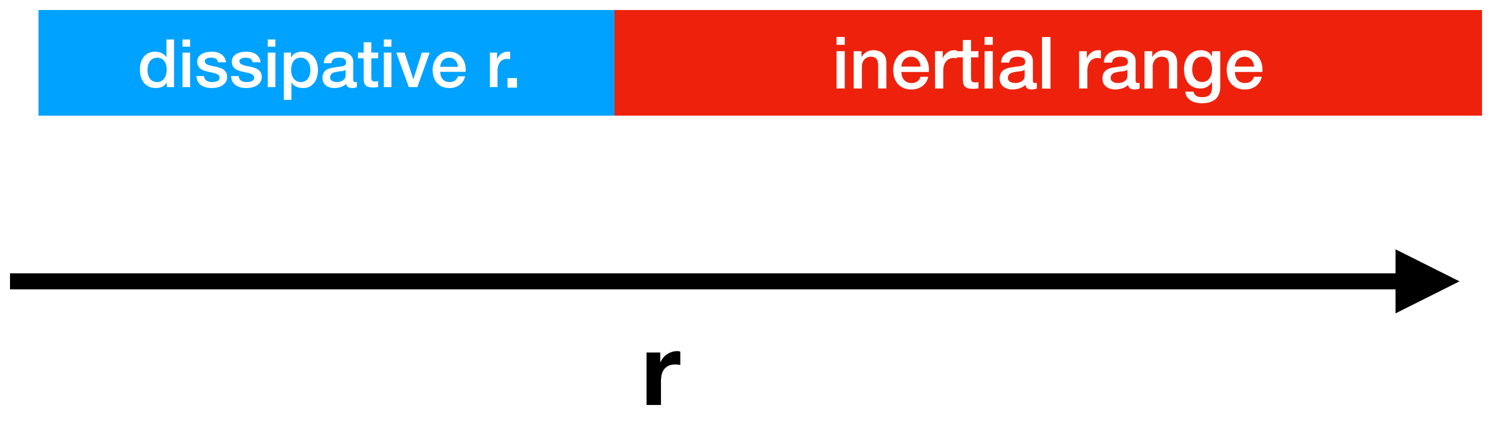
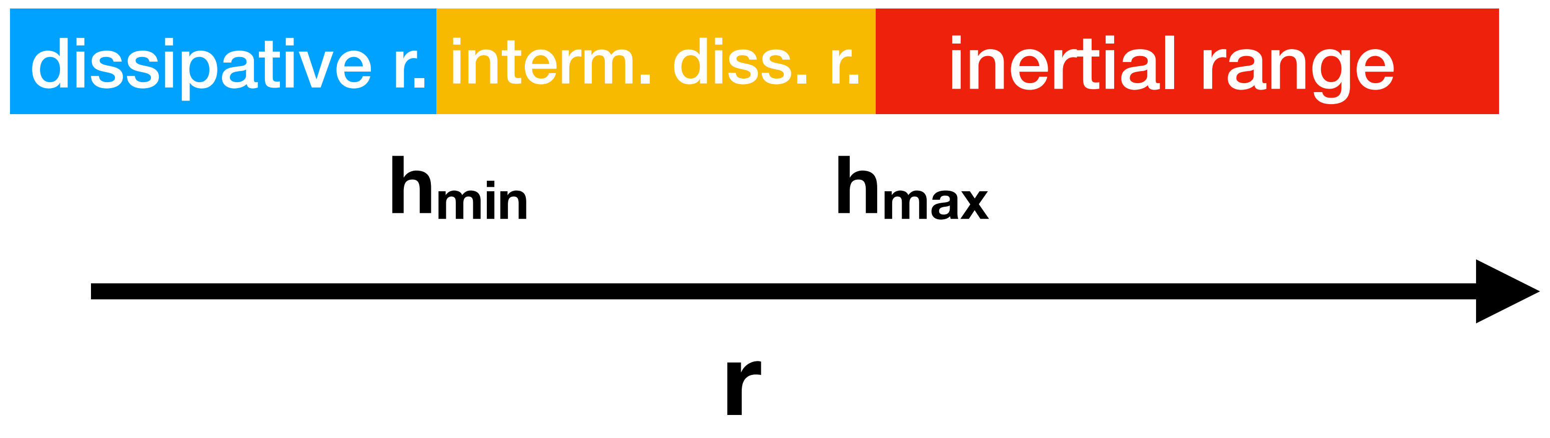


MF

$$\delta_r u \sim \delta_L u \left(\frac{r}{L}\right)^{h}$$

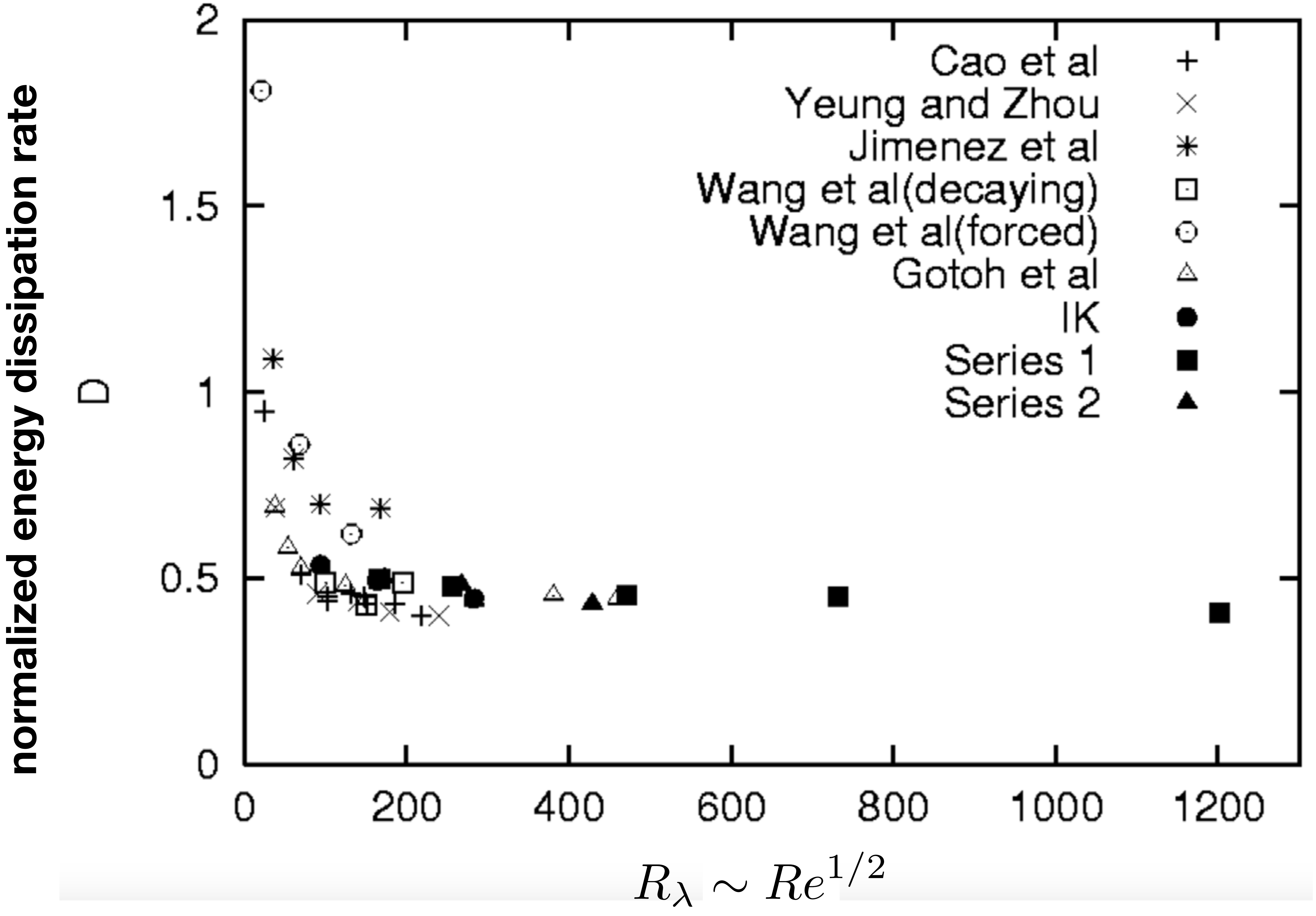
$$1 \approx \frac{\delta_\eta u \eta}{\nu} \approx Re \left(\frac{\eta}{L}\right)^{1+h}$$

$$\left(\frac{\eta}{L}\right) \approx Re^{-1/(1+h)}$$



U. Frisch & M. Vergassola EPL 14, 439 (1991)

zeroth law of turbulence



$$\lim_{\nu \to 0} \nu \langle |\nabla u|^2 \rangle = \epsilon > 0$$

energy dissipation rate independent of viscosity

Fig. 3. The normalised energy dissipation rate, ε (indicated as D on the y-axis), versus R_λ. Results from direct numerical simulations Gotoh et al. (2002), Ishihara & Kaneda (2002), and Kaneda et al. (2003), together with the ones compiled by Sreenivasan (1998), i.e., the data from Cao et al. (1999), Jimenez et al. (1993), Wang et al. (1996), and Yeung & Zhou (1997).
Source: Figure reproduced from [5].

Zero-th law revisited with K41 & MF

K41

$$\epsilon \approx \nu \langle (\nabla u)^2 \rangle$$

$$\left(\frac{\eta}{L}\right) \approx Re^{-3/4}$$

$$\epsilon \approx \nu \left(\frac{\delta_\eta u}{\eta}\right)^2 \sim \nu \left(\frac{\delta_L u (\eta/L)^{1/3}}{L(\eta/L)}\right)^2 \sim \nu \left(\frac{\delta_L u}{L}\right)^2 \left(\frac{\eta}{L}\right)^{-4/3}$$

$$\epsilon \approx \nu \left(\frac{\delta_L u}{L}\right)^2 Re \sim \frac{(\delta_L u)^3}{L}$$

independent of nu

so fits with the zeroth law

Zero-th law revisited with K41 & MF

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independent of nu

so fits with the zeroth law

$$\frac{\eta(h)}{L} \approx Re^{-\frac{1}{(1+h)}} \qquad Prob \sim \left(\frac{\eta(h)}{L}\right)^{3-D(h)} \sim Re^{-\frac{3-D(h)}{(1+h)}}$$

Zero-th law revisited with K41 & MF

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$$\int dh Re^{-\frac{2(h-1)+(3-D(h))}{1+h}} \sim Re^{-\inf\left\{\frac{2(h-1)+(3-D(h))}{1+h}\right\}}$$

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solved by the same h which impose 4/5 law

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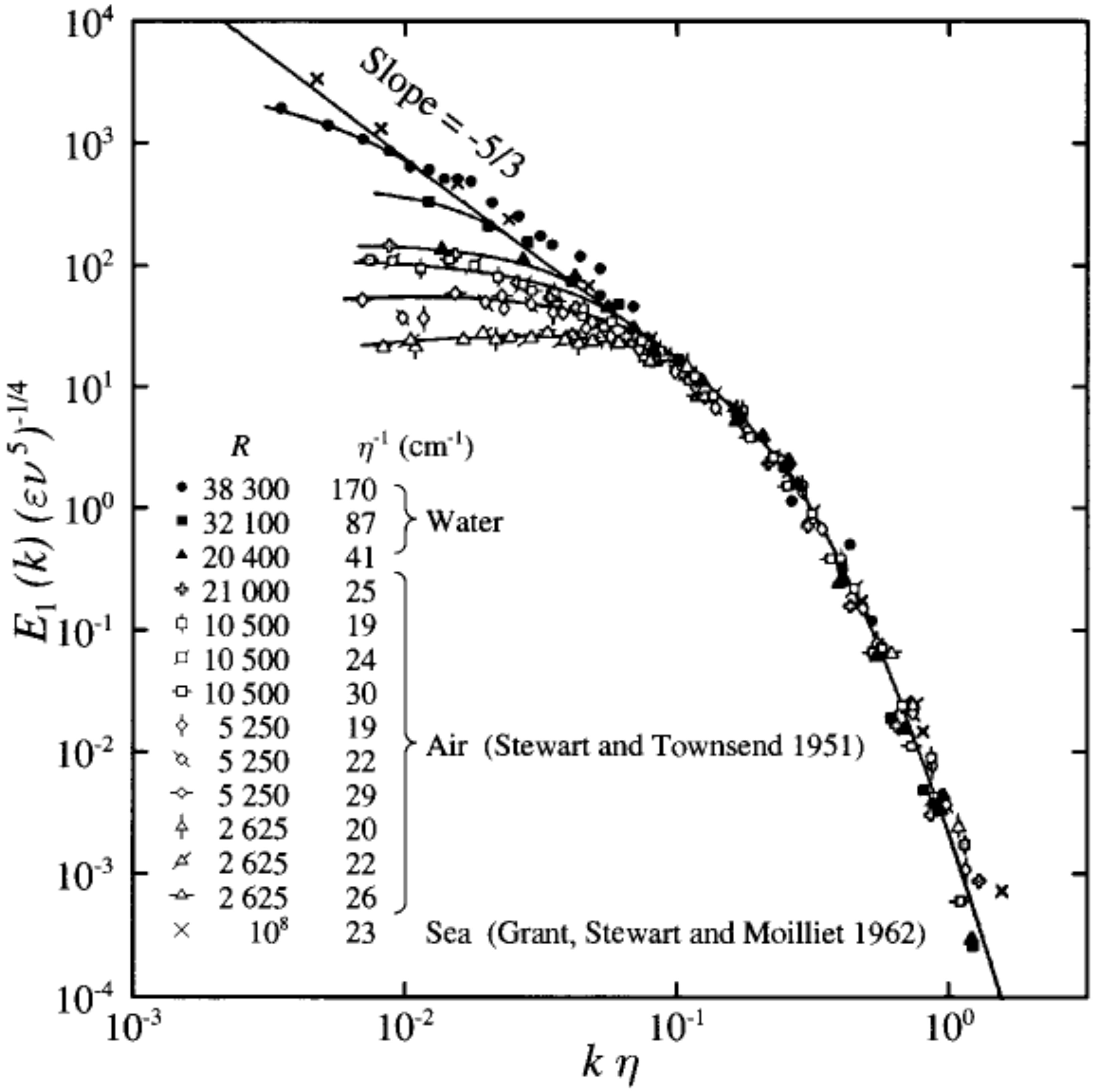
independent of nu

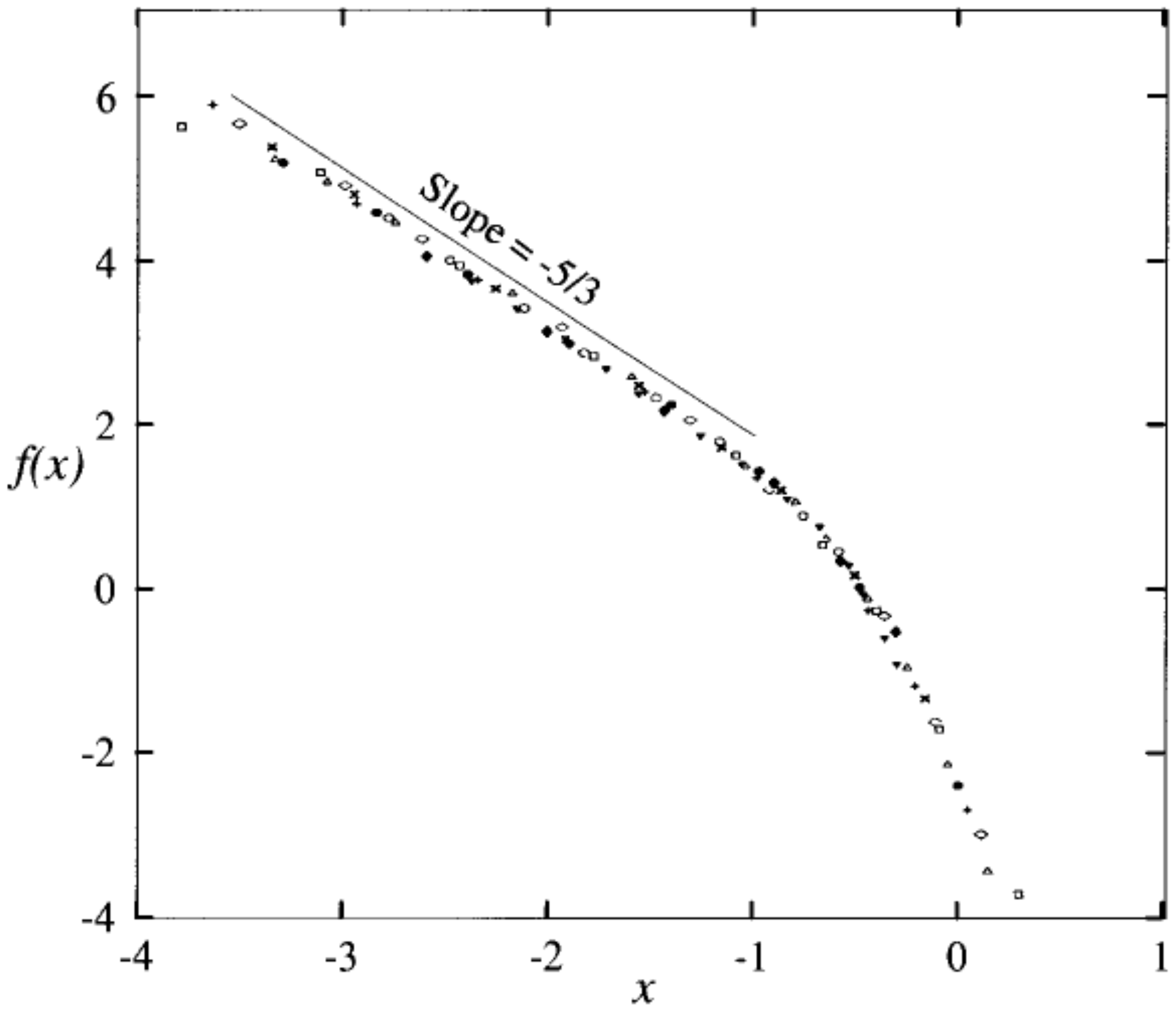
Intermediate dissipative range

rescaling the energy spectra

à la K41



accounting for the intermediate dissipative range



U. Frisch, (1995). *Turbulence: the legacy of AN Kolmogorov*. Cambridge university press.

Shell models: basic ideas

$$\partial_t \hat{\boldsymbol{u}}_{\boldsymbol{k}} = -i\boldsymbol{k}\mathbb{P}(\boldsymbol{k}) \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \hat{\boldsymbol{u}}_{\boldsymbol{p}}\hat{\boldsymbol{u}}_{\boldsymbol{q}} - \nu k^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \hat{\boldsymbol{F}}_{\boldsymbol{k}}$$

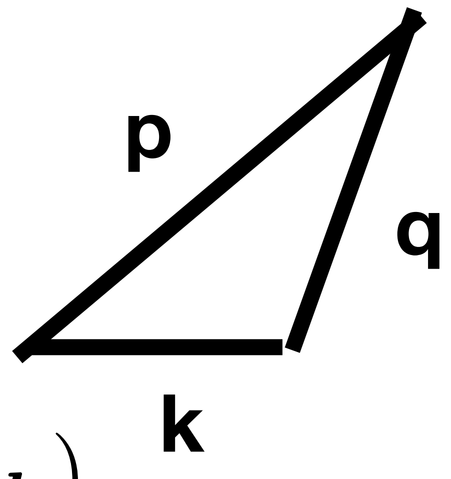
$$\boxed{\nu, F \to 0}$$

conserved

Energy $E = \frac{1}{2}\int d\boldsymbol{x} u^2 = \frac{1}{2}\sum_{\boldsymbol{k}} |\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2$ **3D**

Helicity $H = \frac{1}{2}\int d\boldsymbol{x}\, \boldsymbol{u}\cdot\boldsymbol{\omega} = \frac{1}{2}\sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}\cdot\hat{\boldsymbol{\omega}}_{-\boldsymbol{k}}$ **3D**

Conservations holds triad by triad

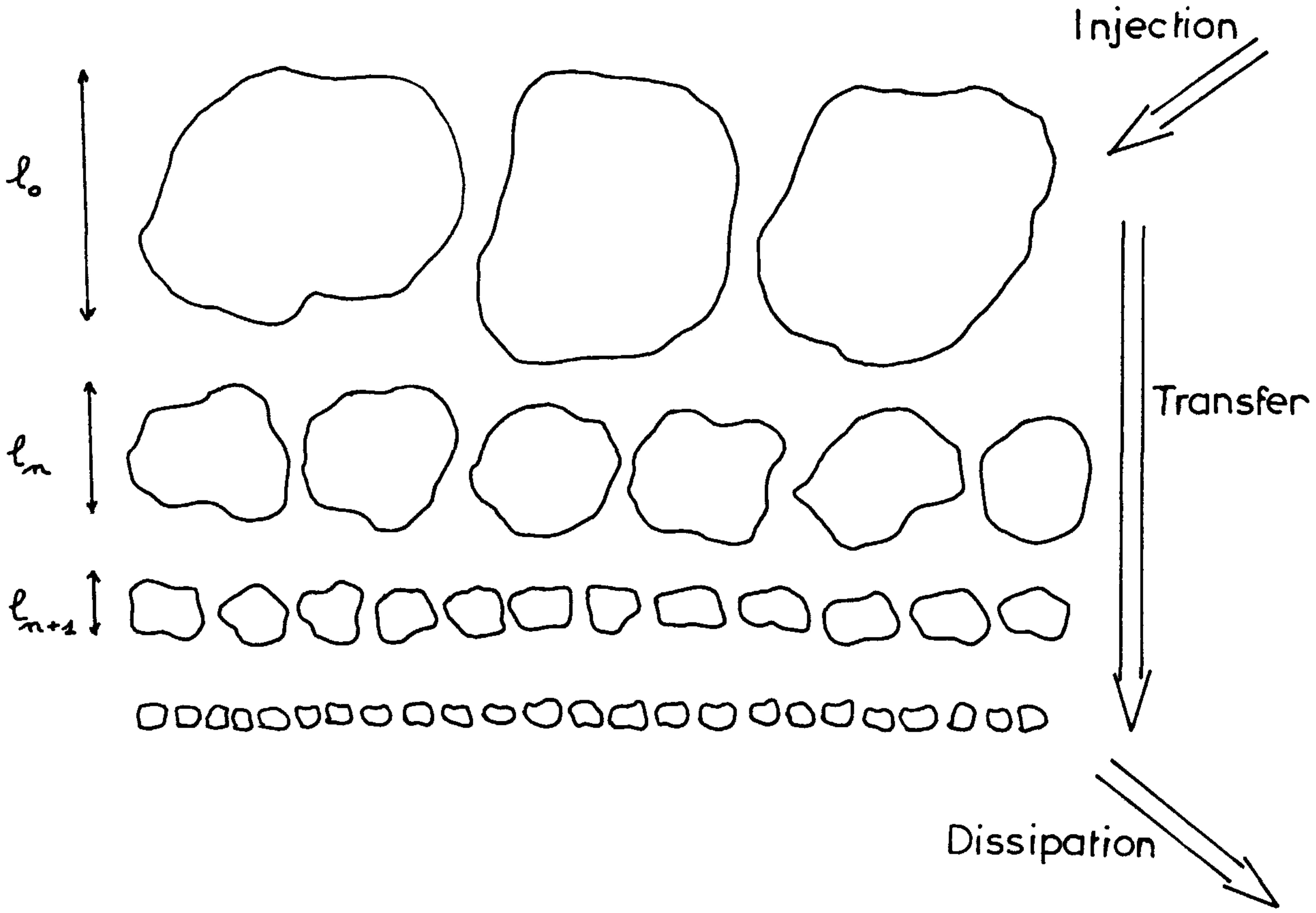


$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$

phase space preserved $\frac{\partial(\partial_t \hat{\boldsymbol{u}}_{\boldsymbol{k}})}{\partial \hat{\boldsymbol{u}}_{\boldsymbol{k}}} = 0$

+ locality

IDEA: we keep discrete representative scales $\ell_n = \ell_0\lambda^{-n}$
with a representative velocity variable at each scale
so to focus on the energy cascade process only
and define a "reduced" dynamics to mimic the energy cascade



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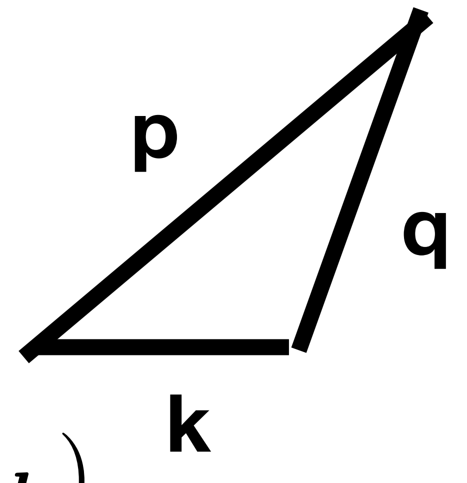
Helicity $H = \frac{1}{2}\int d\boldsymbol{x}\, \boldsymbol{u}\cdot\boldsymbol{\omega} = \frac{1}{2}\sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{\omega}}_{-\boldsymbol{k}}$ **3D**

Conservations holds triad by triad



$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$

phase space preserved $\dfrac{\partial(\partial_t \hat{\boldsymbol{u}}_{\boldsymbol{k}})}{\partial \hat{\boldsymbol{u}}_{\boldsymbol{k}}} = 0$

+ locality

$$\frac{du_n}{dt} = k_n G_n[\boldsymbol{u},\boldsymbol{u}] - \nu k_n^2 u_n + f_n$$

- $k_n = k_0\lambda^n$ logarithmically equispaced shells (typically $\lambda = 2$)
- G_n quadratic in u of the form $u_{n'}u_{n''}$
- for $\nu = f_n = 0$ it should preserve the **quadratic invariants** of NS e.g.
  $E = 1/2\sum_n |u_n|^2 \quad H = 1/2\sum_n (-1)^n k_n |u_n|^2$
- for $\nu = f_n = 0$ it should preserve them **triad by triad**
- for $\nu = f_n = 0$ phase space is preserved
- **locality** (non needed in principle) interactions among shells are local (n',n" are close to n)

NB: the above requests do not fix unambiguously G_n

PROS: extended inertial range with few d.o.f. , absence of sweeping, ideal framework to study the cascade of energy
CONS: absence of geometry (can only be partially cured), but this can also be a pros as shown later

Sabra shell model

SABRA model (L'vov et al 1998)

$$\frac{du_n}{dt} = i\left(ak_{n+1}u_{n+2}u_{n+1} + bk_n u_{n+1}u_{n-1} + ck_{n-1}u_{n-1}u_{n-2}\right)^* - \nu k_n^2 u_n + f_n \qquad n = 0, N$$

$$E = 1/2 \sum_n |u_n|^2$$

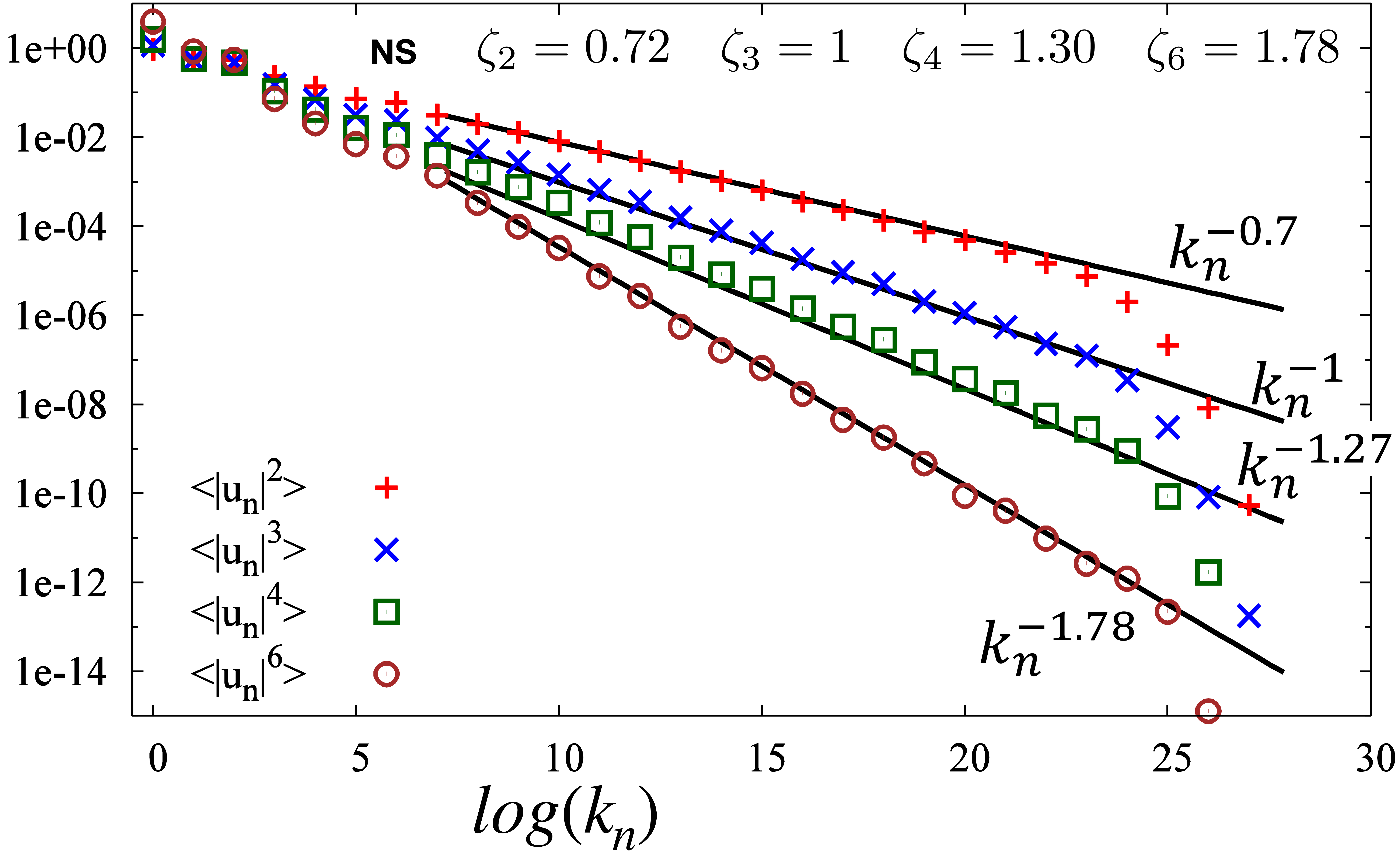
$$H = 1/2 \sum_n (-1)^n k_n |u_n|^2$$

$$\dot{E} = 0 \implies a + b + c = 0$$

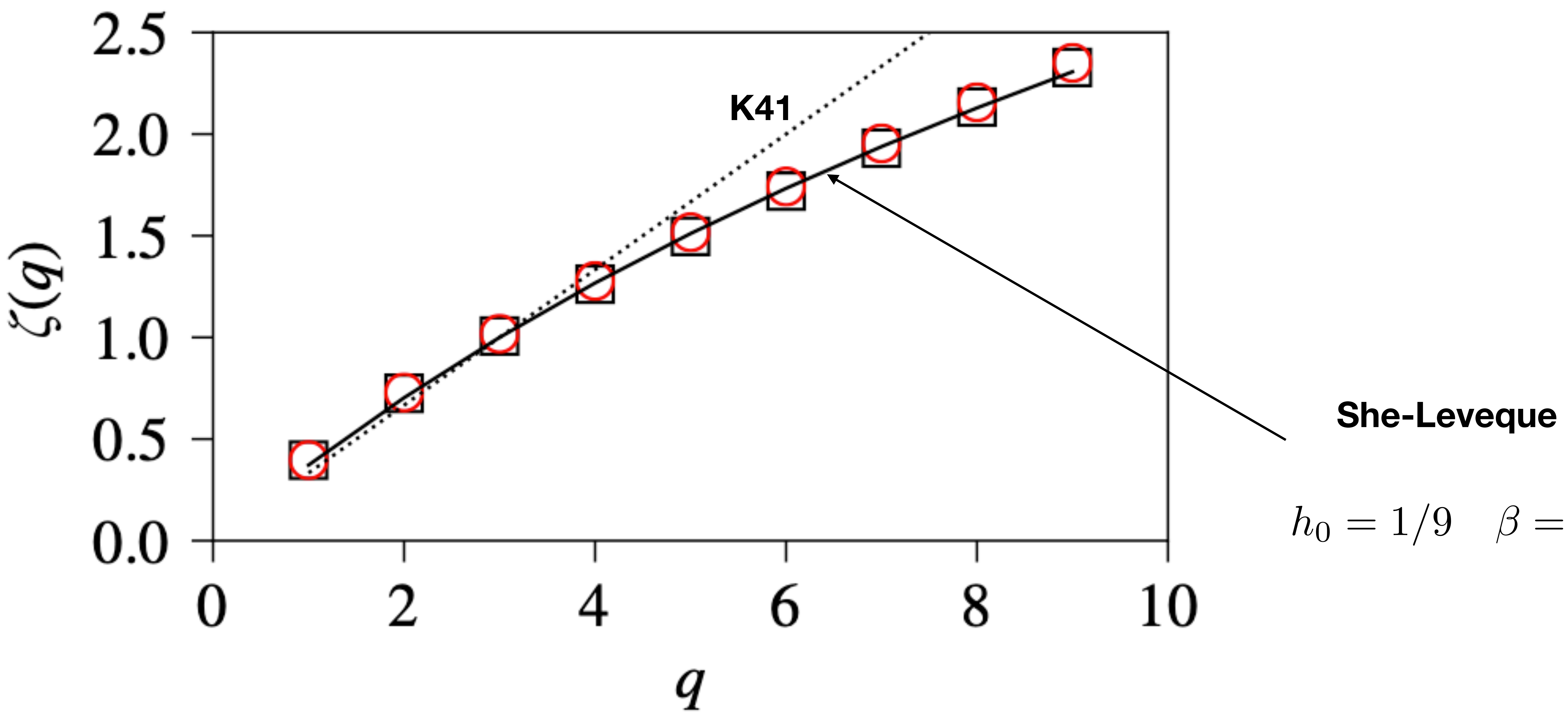
$$\dot{H} = 0 \implies \frac{a}{c} = -1$$

$$a = 1 \quad b = c = -\frac{1}{2}$$

$$u_{-1} = u_{-2} = u_{N+1} = u_{N+2} = 0$$



Shell model & MF



$h_0 = 1/9 \quad \beta = 0.6$

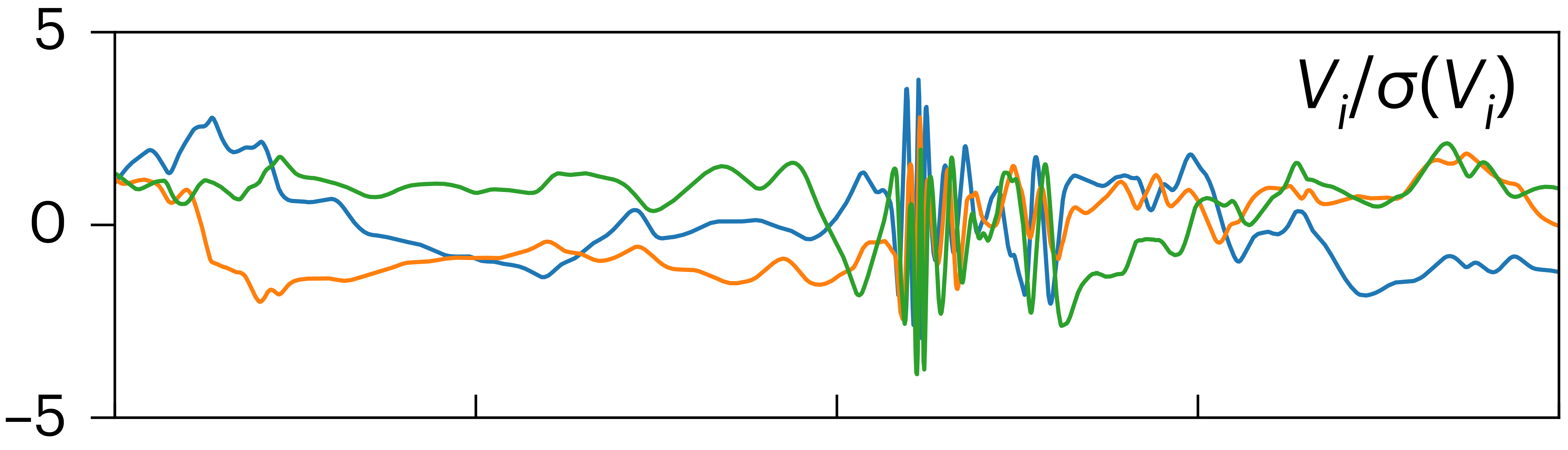
$$D(h) = \frac{3(h-h_0)}{\log(\beta)}\left[\log\left(\frac{3(h_0-h)}{d_0\log(\beta)}\right)-1\right]+3-d_0 \qquad \zeta_p = h_0 p + \frac{(1-3h_0)}{(1-\beta)}(1-\beta^{p/3})$$

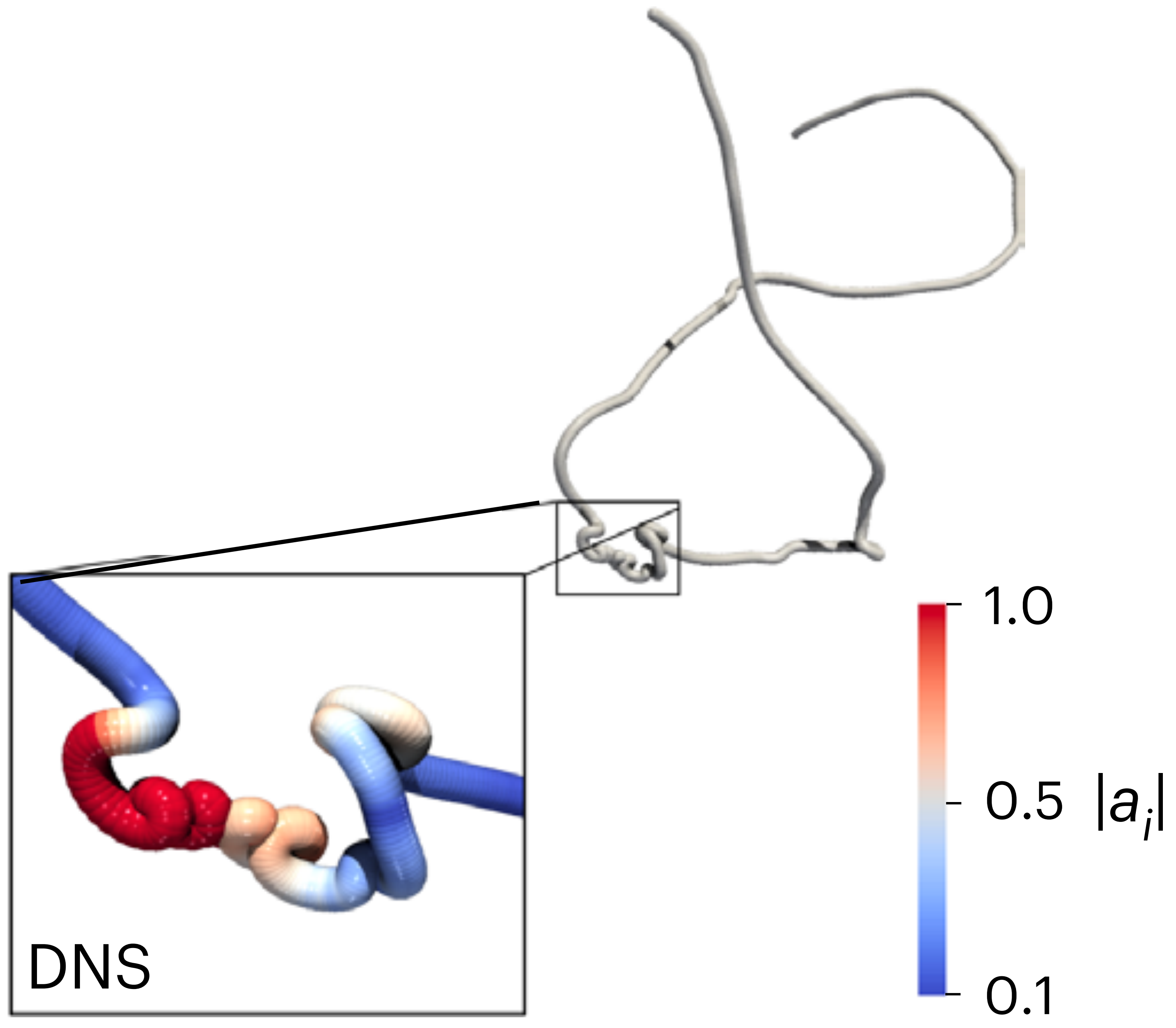
Lagrangian turbulence

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t)$$



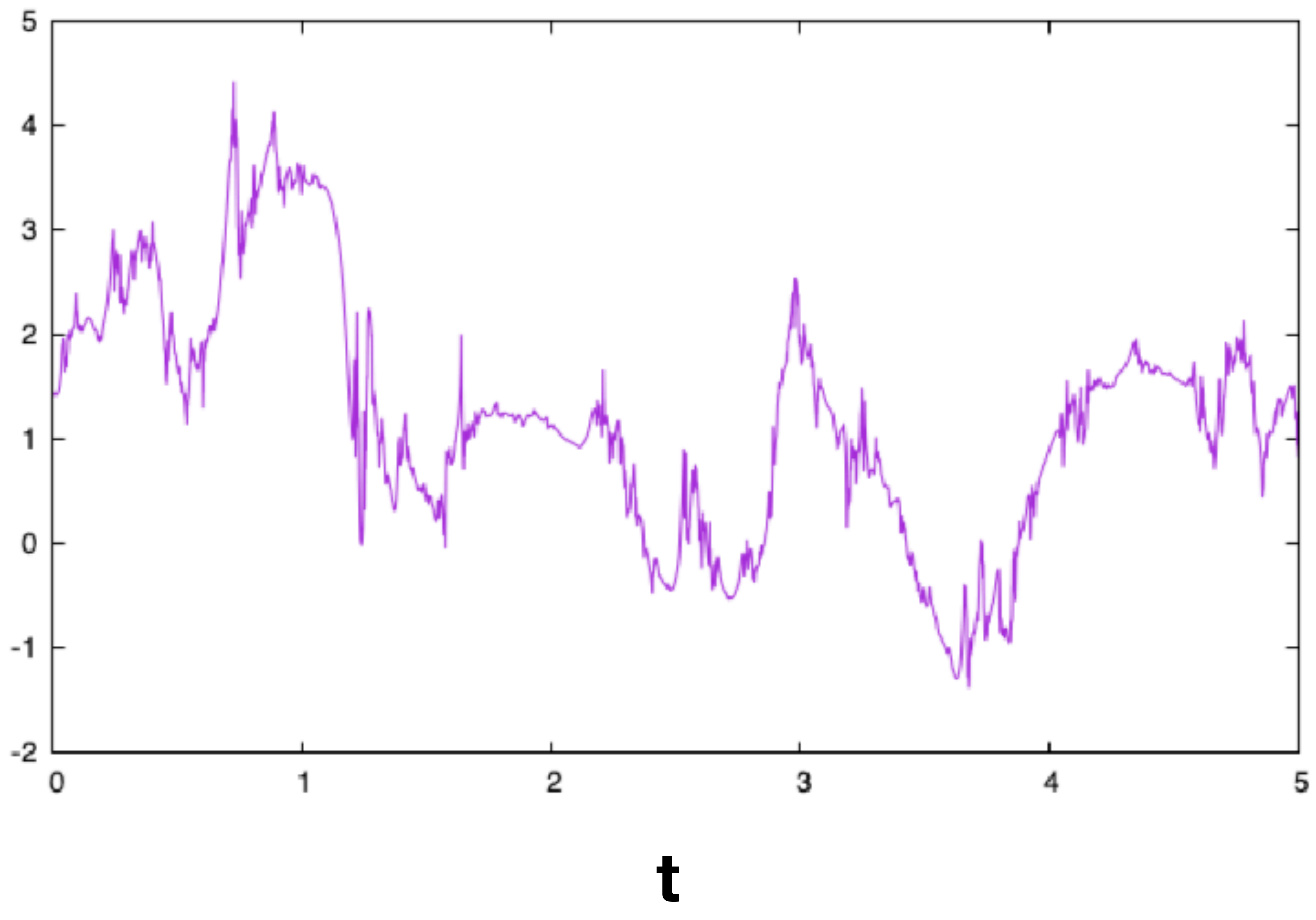


We learned about the statistical properties of the velocity field in the Eulerian frame and to recognize the signature of intermittency and the cascade what about the Lagrangian frame?

Shell model for Lagrangian turbulence

$$\left(d/dt + \nu k_n^2\right)u_n = i\left(k_n u_{n+2}u_{n+1}^* - b k_{n-1}u_{n+1}u_{n-1}^* - c k_{n-2}u_{n-1}u_{n-2}\right) + f_n.$$

$$v(t) \equiv \sum_{n=1}^{N} \mathrm{Re}(u_n)$$



Sum of components spanning a large range of scales and characterized by many characteristic times

a simple way to build

PHYSICAL REVIEW E **66**, 066307 (2002)

Lagrangian statistics and temporal intermittency in a shell model of turbulence

G. Boffetta, F. De Lillo, and S. Musacchio
Dipartimento di Fisica Generale and INFM, Università di Torino, Via Pietro Giuria 1, 10125 Torino, Italy
and CNR-ISAC–Sezione di Torino, Corso Fiume 4, 10133 Torino, Italy
(Received 16 July 2002; published 19 December 2002)

Lagrangian structure functions

$$S_p(r) = \langle [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - u(\boldsymbol{x},t)\cdot\hat{\boldsymbol{r}}]^p\rangle = \langle(\delta_r u)^p\rangle \sim u_L^p \left(\frac{r}{L}\right)^{\zeta_p} \qquad \eta \ll r \ll L$$

$$S_p^L(r) = \langle [u(\boldsymbol{X}(t+\tau),t+\tau) - u(\boldsymbol{X}(t),t)]^p\rangle = \langle(\delta_\tau u)^p\rangle \sim u_L^p \left(\frac{\tau}{T_L}\right)^{\xi_p} \qquad \tau_\eta \ll \tau \ll T_L$$

$$\tau_\eta = \frac{\eta}{\delta_\eta u} \qquad T_L = \frac{L}{u_L} \qquad \frac{T_L}{\tau_\eta} \approx Re_\lambda \approx Re^{1/2}$$

Lagrangian structure functions

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$$\tau_\eta = \frac{\eta}{\delta_\eta u} \qquad T_L = \frac{L}{u_L} \qquad \frac{T_L}{\tau_\eta} \approx Re_\lambda \approx Re^{1/2}$$

Dimensional argument à la K41

$$[\epsilon] = [U]^2[T]^{-1} \quad \delta_\tau u \sim (\epsilon\tau)^{1/2} \quad \Rightarrow \quad S_p^L(\tau) = \langle(\delta_\tau u)^p\rangle \sim (\epsilon\tau)^{p/2}$$

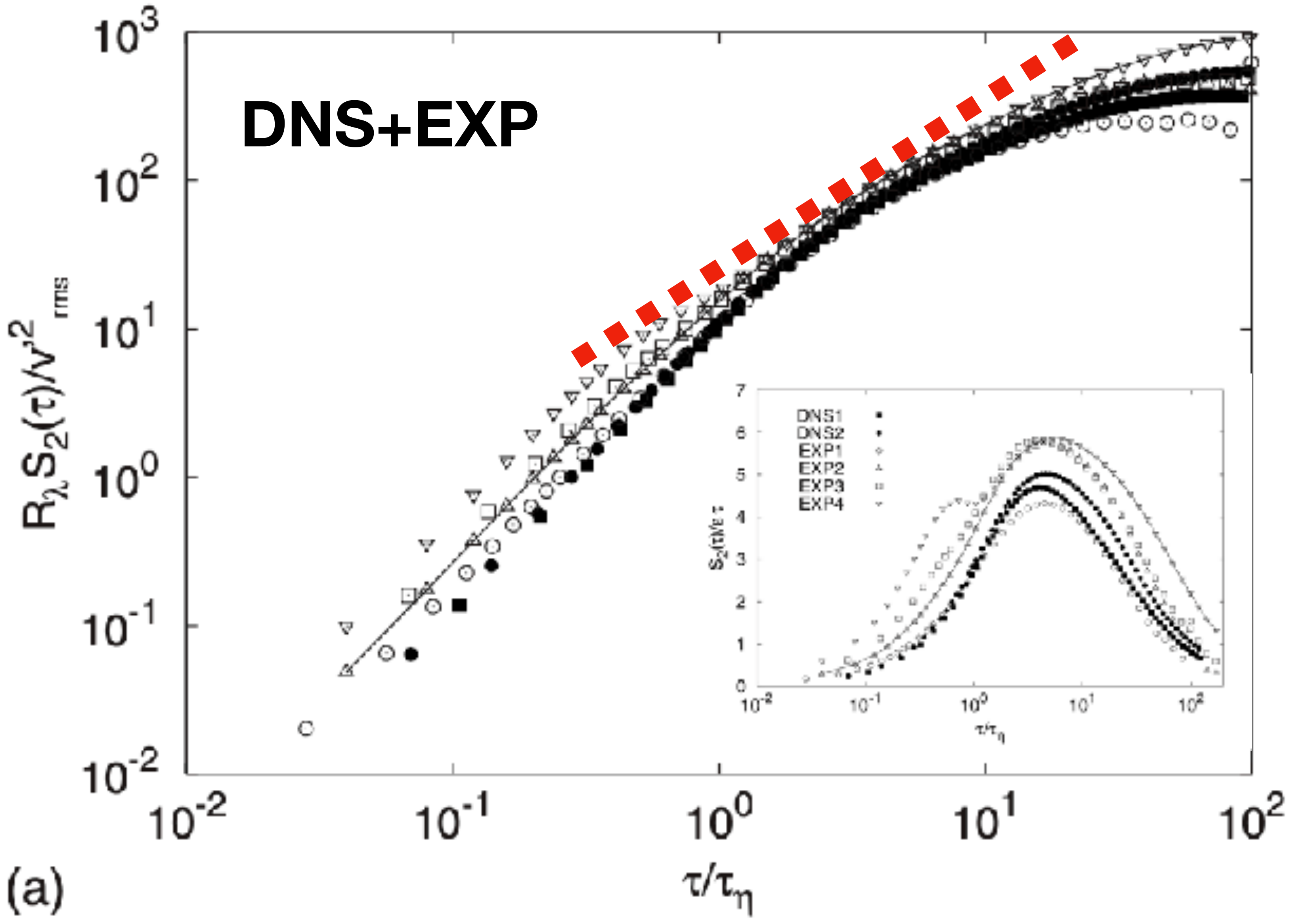
(DNS+EXP) L. Biferale, E. Bodenschatz, M. Cencini, AS. Lanotte, NT. Ouellette, F Toschi, and H Xu. *Physics of Fluids* 20, no. 6 (2008)

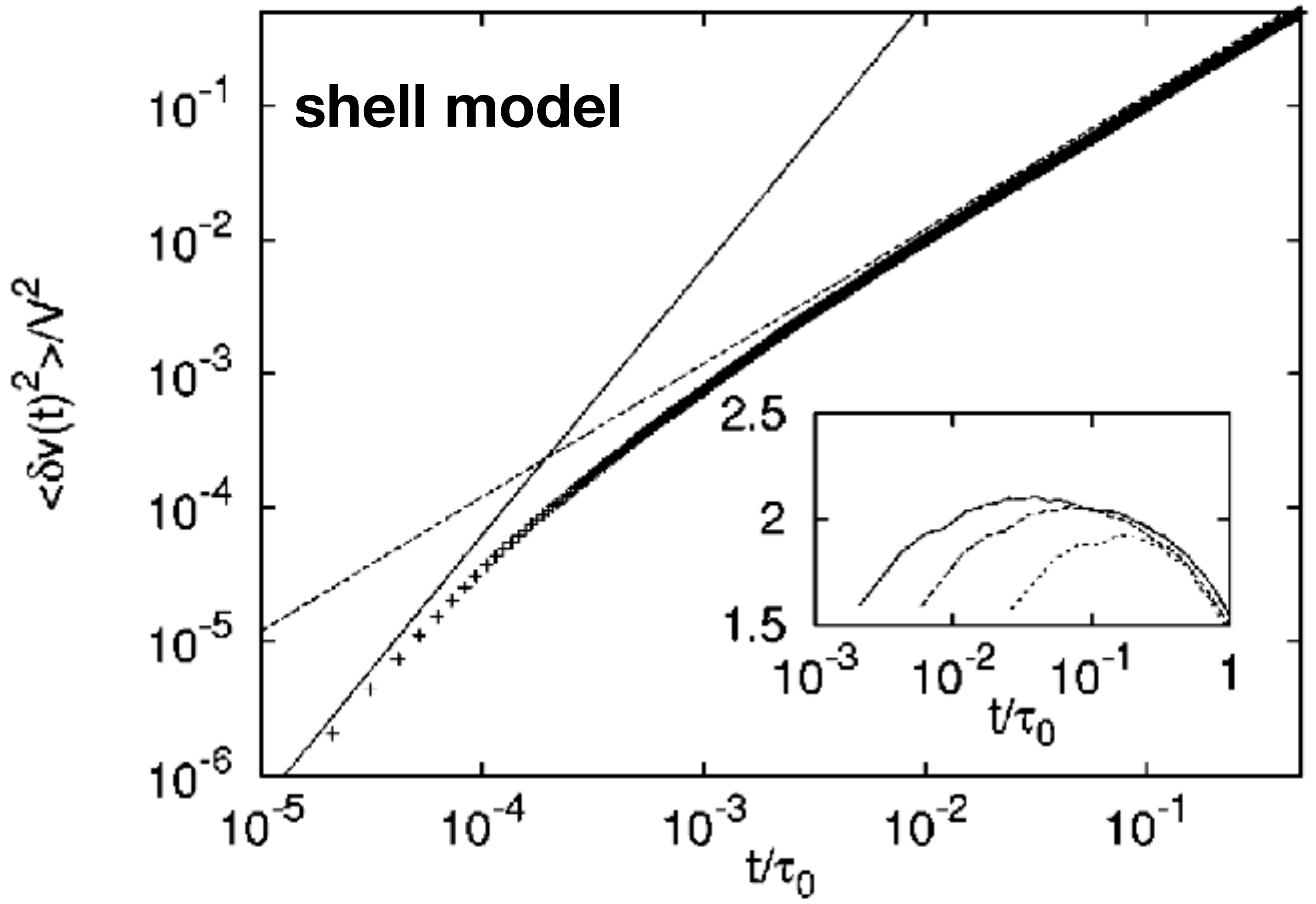
(Shell model) G. Boffetta, F. De Lillo, S. Musacchio.. *PRE* 66, 066307 (2002)

see also G. Falkovich, H Xu, A Pumir, E Bodenschatz, L Biferale, G Boffetta, AS Lanotte, F Toschi. *Physics of Fluids* 24, no. 5 (2012).

Is there intermittency as for the Eulerian SF?

Are ξ_q related with ζ_q? If yes how?



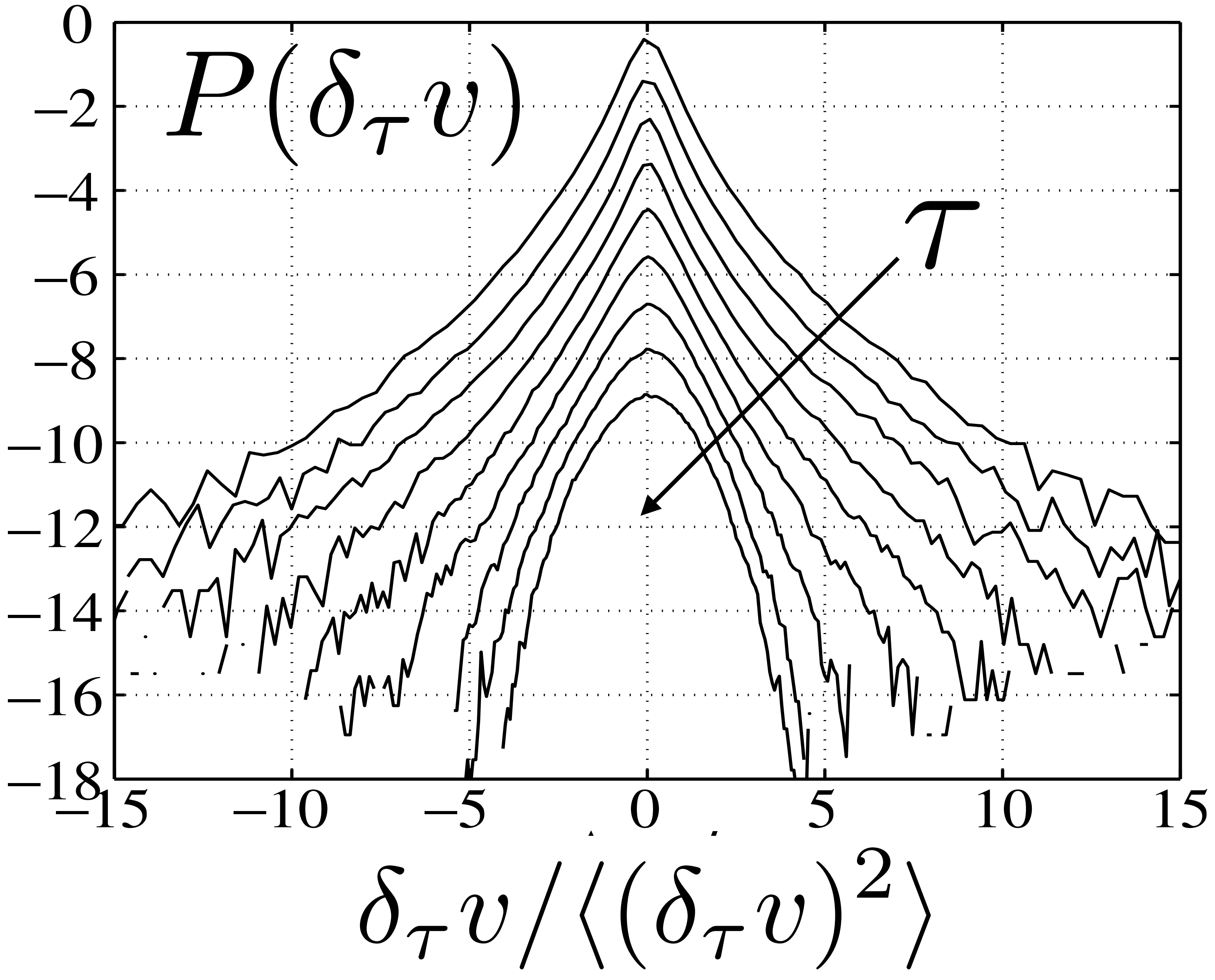


Lagrangian Structure function and intermittency

Experiments, DNS and shell models show that Lagrangian velocity increments are highly intermittent

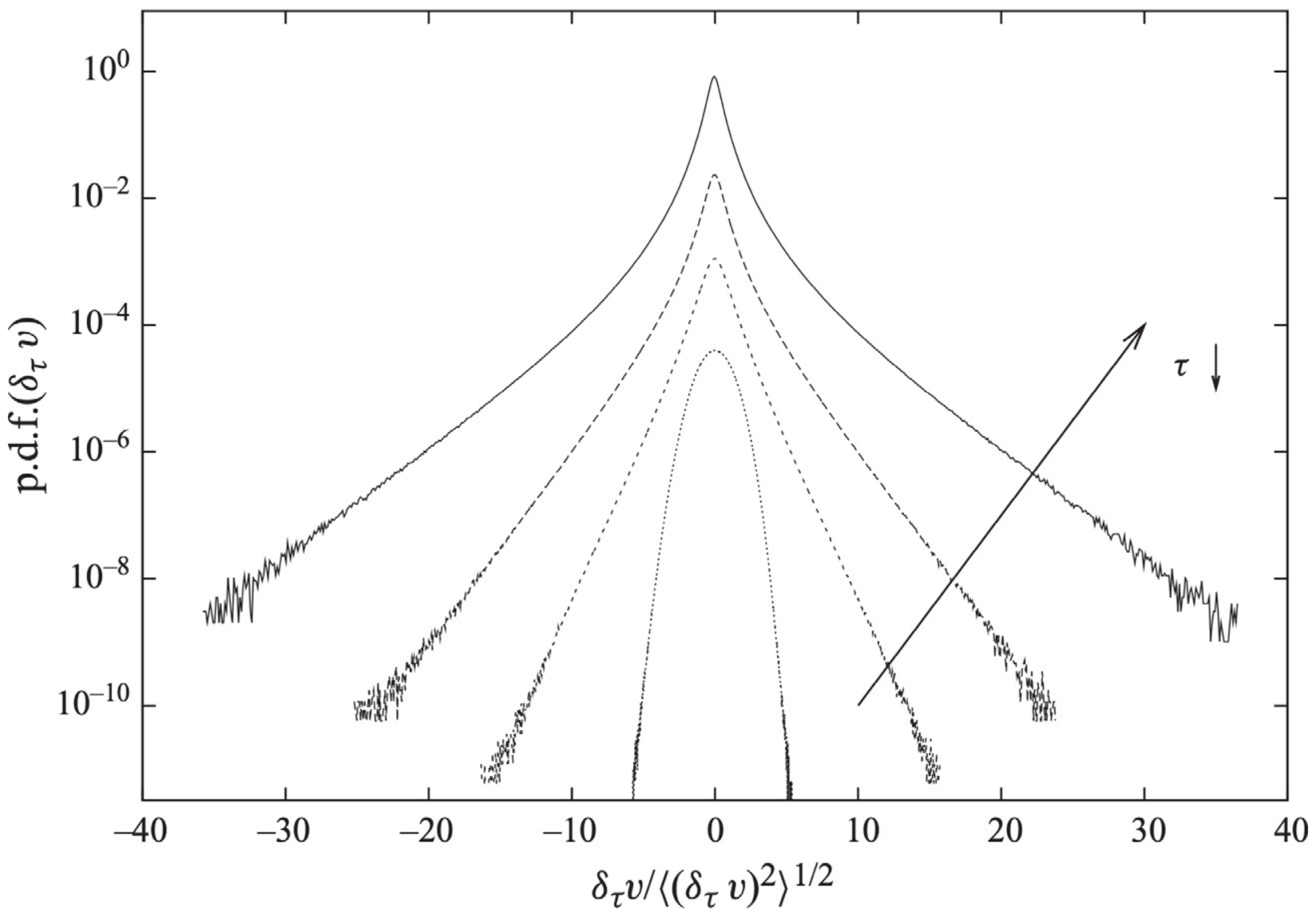
(Mordant et al PRL 2001)

EXP



(R. Benzi, et al JFM 653 , 221 (2010))

DNS



(G. Boffetta et al PRE 66, 066307 (2002))

Shell model

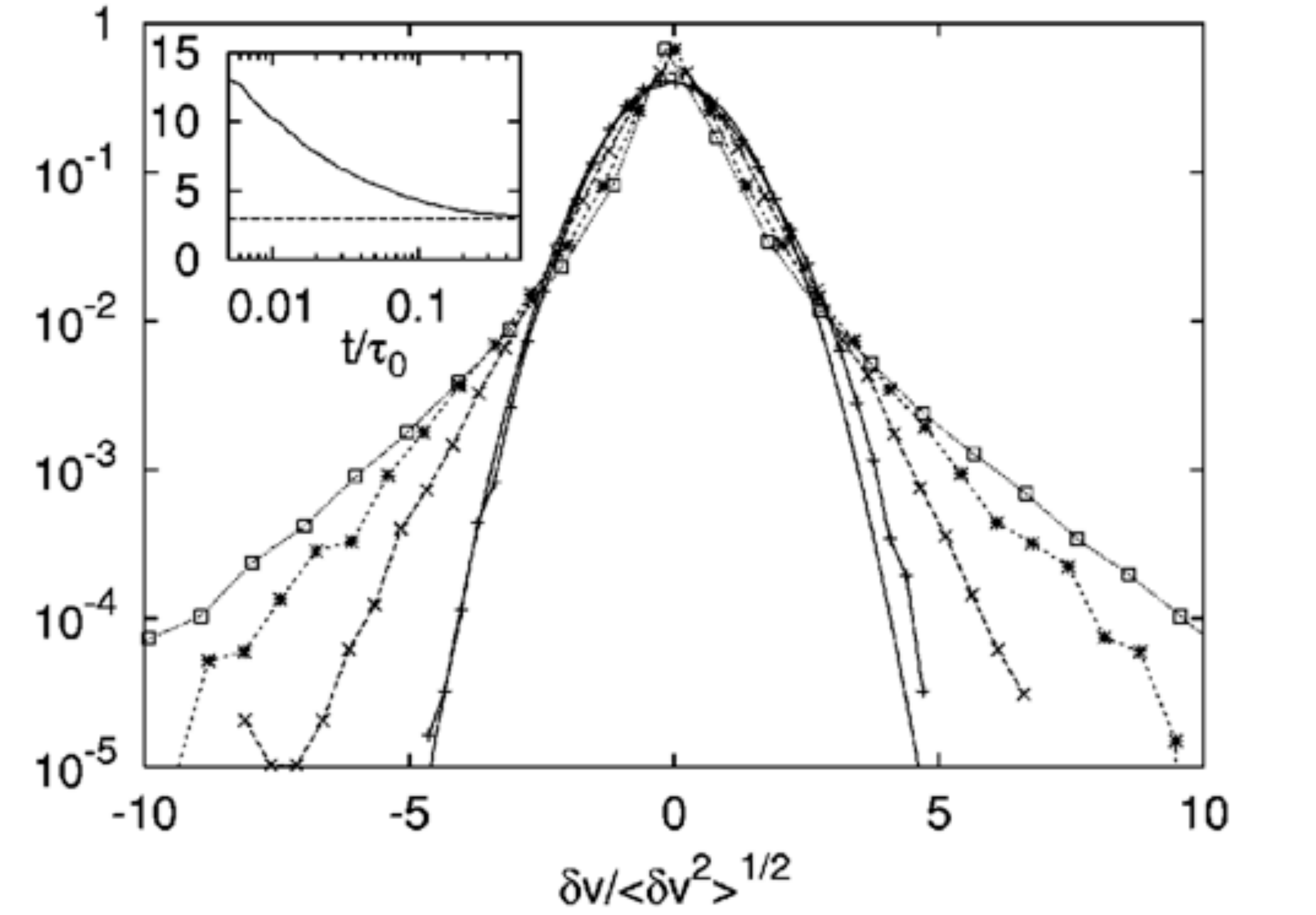


FIG. 3. Probability density functions of velocity differences $\delta v(t)$ normalized with the variance at time lags $t/\tau_0 = 0.002$ (□), 0.01 (*), 0.06(×), 0.35(+). The continuous line represents a Gaussian. Inset: flatness $F = \langle \delta v(t)^4 \rangle / \langle \delta v(t)^2 \rangle^2$ as a function of time and Gaussian value $F = 3$ (dashed line).

Lagrangian Structure functions

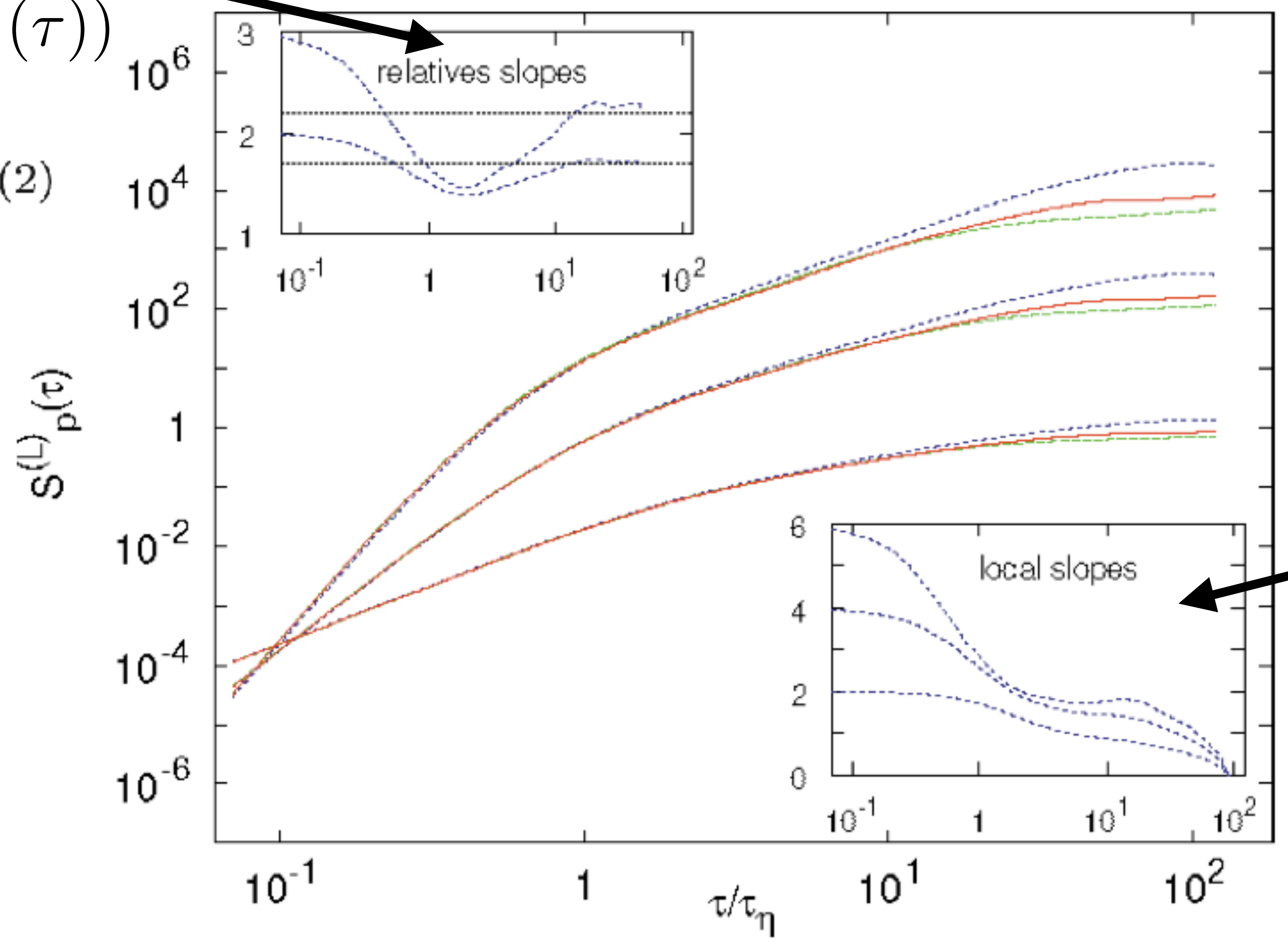
(Mordant et al PRL 2001)

$$\frac{d\ln(S_p^L(\tau))}{d\ln(S_2^L(\tau))}$$

=

$$\xi^{(p)}/\xi^{(2)}$$

(L. Biferale et al, *JoT* 7, N6 (2006):)



$$\frac{d\ln(S_p^L(\tau))}{d\ln(\tau)}$$

Figure 6. Log-log plot of Lagrangian structure functions of orders $p = 2, 4, 6$ (bottom to top) versus τ, along the three spatial directions. At any order p, from top to bottom we have the x, z, y components, respectively. In the inset on the bottom right, logarithmic local slopes of all orders are shown for the x-component, the most energetic one. In the inset on the top-left, relative local slopes, with respect to the second order, are shown $\mathrm{d}\log S_p(\tau)/\mathrm{d}\log S_2(\tau)$ for $p = 4, 6$ and the x-component, together with the multifractal predictions (line). Data refer to $R_\lambda = 284$.

Scaling is typically very poor that the best is to use Extended Self Similarity plots i.e.

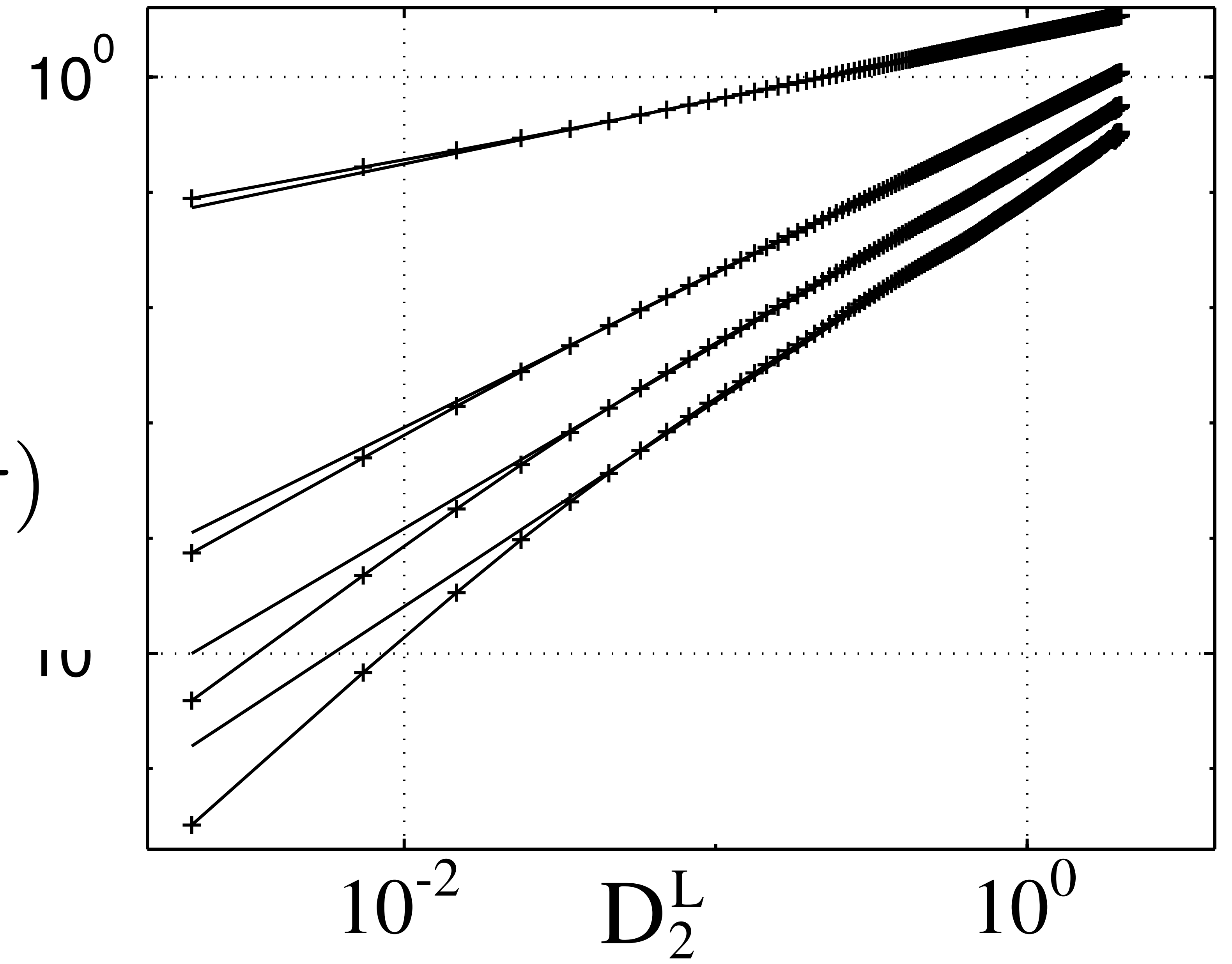
$$S_p^L(\tau) \quad vs \quad S_2^L(\tau)$$

Why scaling is so bad?

Lagrangian Structure functions

(Mordant et al PRL 2001)

$S_q^L(\tau)$

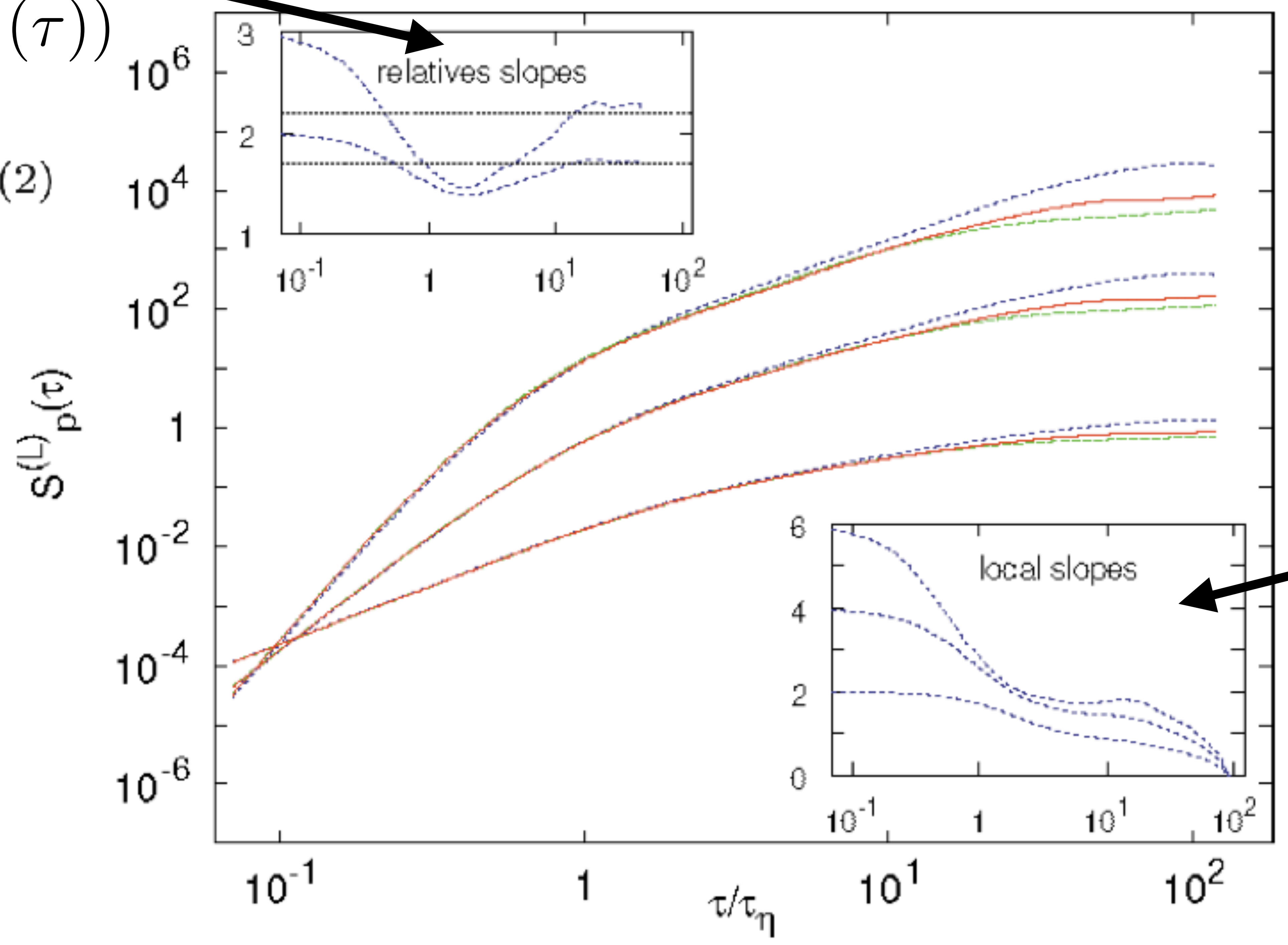


$S_2^L(\tau)$

FIG. 5. ESS plots of the s ction variation (in double log coordinates). The solid curves are best linear fits with slopes equal to $\xi_q^L = 0.56 \pm 0.01$, 1.34 ± 0.02, 1.56 ± 0.06, and 1.8 ± 0.2 for $p = 1, 3, 4$, and 5 from top to bottom. Coordinates in arbitrary units.

$$\frac{d\ln(S_p^L(\tau))}{d\ln(S_2^L(\tau))} = \xi^{(p)}/\xi^{(2)}$$

(L. Biferale et al, *JoT* 7, N6 (2006):)



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Why scaling is so bad?

MF model for Lagrangian turbulence

Eulerian MF

$$\delta u(r) \sim u_L \left(\frac{r}{L}\right)^h \qquad \langle(\delta u(r))^p\rangle \sim u_L^p \int dh \left(\frac{r}{L}\right)^{hp+3-D(h)} \qquad \zeta(p) = \min_h \{[ph + 3 - D(h)]\}$$

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Bridging Lagrangian and Eulerian MF (Borgas 1993, Boffetta et al 2002, Chevillard et al 2003)

$$\delta v(\tau) \sim \delta u(r) \quad \text{if} \quad \tau \sim \frac{r}{\delta u(r)} \Rightarrow \tau \sim \frac{r}{u_L (r/L)^h} \xrightarrow[T = \frac{L}{u_L}]{} \left(\frac{r}{L}\right) \sim \left(\frac{\tau}{T}\right)^{\frac{1}{1-h}}$$

MF model for Lagrangian turbulence

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MF in the Lagrangian framework

$$\langle(\delta v(\tau))^p\rangle \sim u_L^p \int dh \left(\frac{\tau}{T}\right)^{\frac{hp+3-D(h)}{1-h}} \sim \left(\frac{\tau}{T}\right)^{\xi(p)} \qquad \xi(p) = \min_h \left\{\frac{ph + 3 - D(h)}{1-h}\right\}$$

NB: according to MF model the same D(h) rules both Eulerian and Lagrangian statistics, hence **if the predictions with the D(h) that works for Eulerian SF works also in the Lagrangian frame, it is a stringent test of MF!**

A comment on the intermediate dissipative range

MF- Eulerian

$$\delta_r u \sim \delta_L u \left(\frac{r}{L}\right)^h$$

$$1 \approx \frac{\delta_\eta u \eta}{\nu} \approx Re \left(\frac{\eta}{L}\right)^{1+h}$$

$$\left(\frac{\eta}{L}\right) \approx Re^{-1/(1+h)}$$

A comment on the intermediate dissipative range

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MF- Lagrangian

$$\tau_\eta \sim \frac{\eta}{\delta_\eta u} \sim \frac{\eta}{\delta_L u (\eta/L)^h} = \frac{L}{\delta_L u} \left(\frac{\eta}{L}\right)^{1-h}$$

A comment on the intermediate dissipative range

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MF- Lagrangian

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$$T_L \sim \frac{L}{\delta_L u}$$

$$\boxed{\left(\frac{\tau_\eta}{T_L}\right) \sim Re^{-\frac{1-h}{1+h}}}$$

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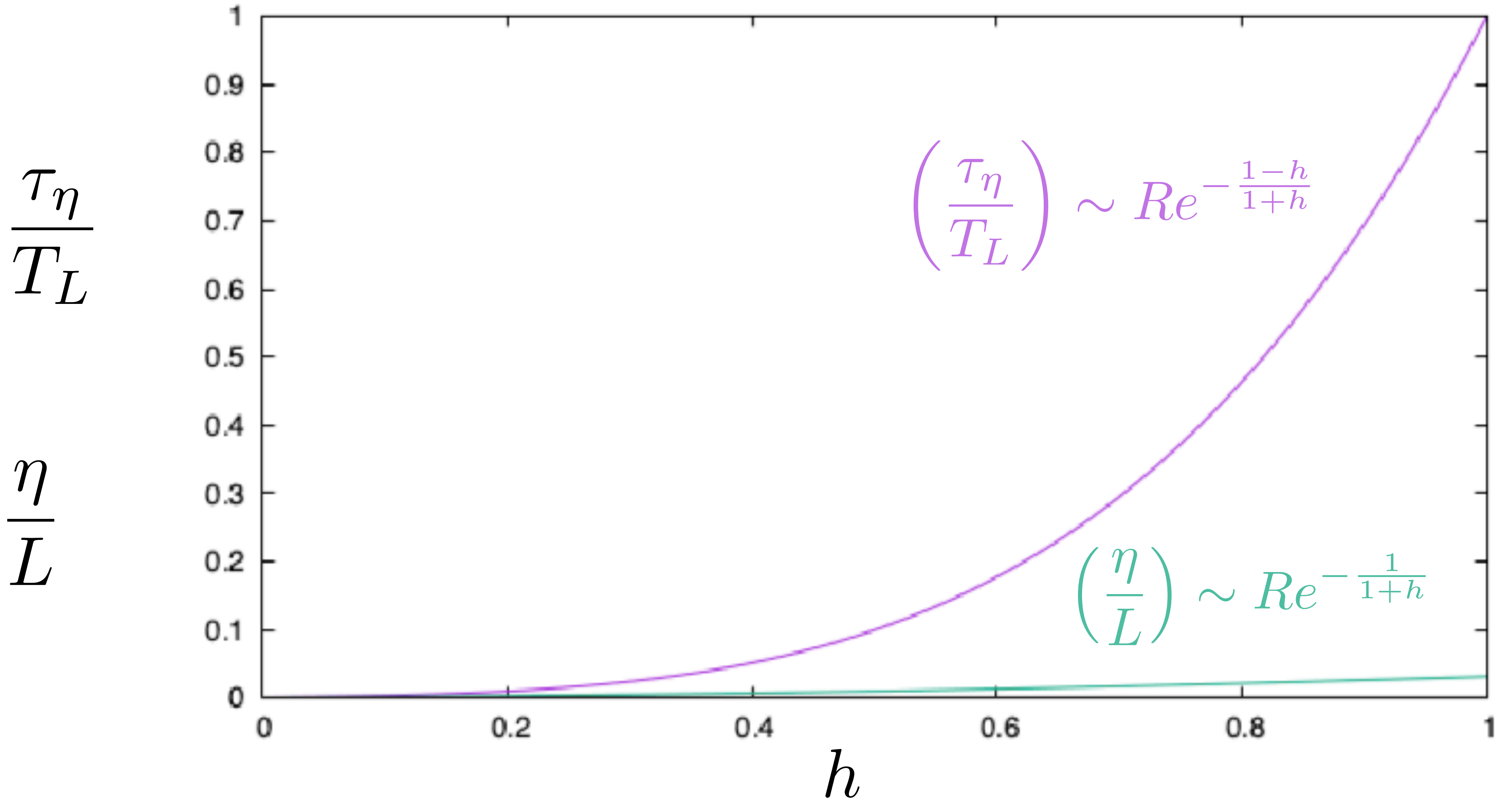
$$\left(\frac{\tau_\eta}{T_L}\right) \sim Re^{-\frac{1-h}{1+h}}$$



A consequence of intermediate dissipative range of MF model is that **Lagrangian statistics** is expected to be much **more contaminated by fluctuations of the dissipative time scale** than Eulerian statistics

Which explains the poor scaling!

what does happen removing intermittency

$$\frac{du_n}{dt} = ik_n\left(u_{n+2}u^*_{n+1} - \frac{\delta}{2}u_{n+1}u^*_{n-1} + \frac{1-\delta}{4}u_{n-1}u_{n-2}\right) - \nu k_n^2 u_n + f_n \,, \qquad (5)$$

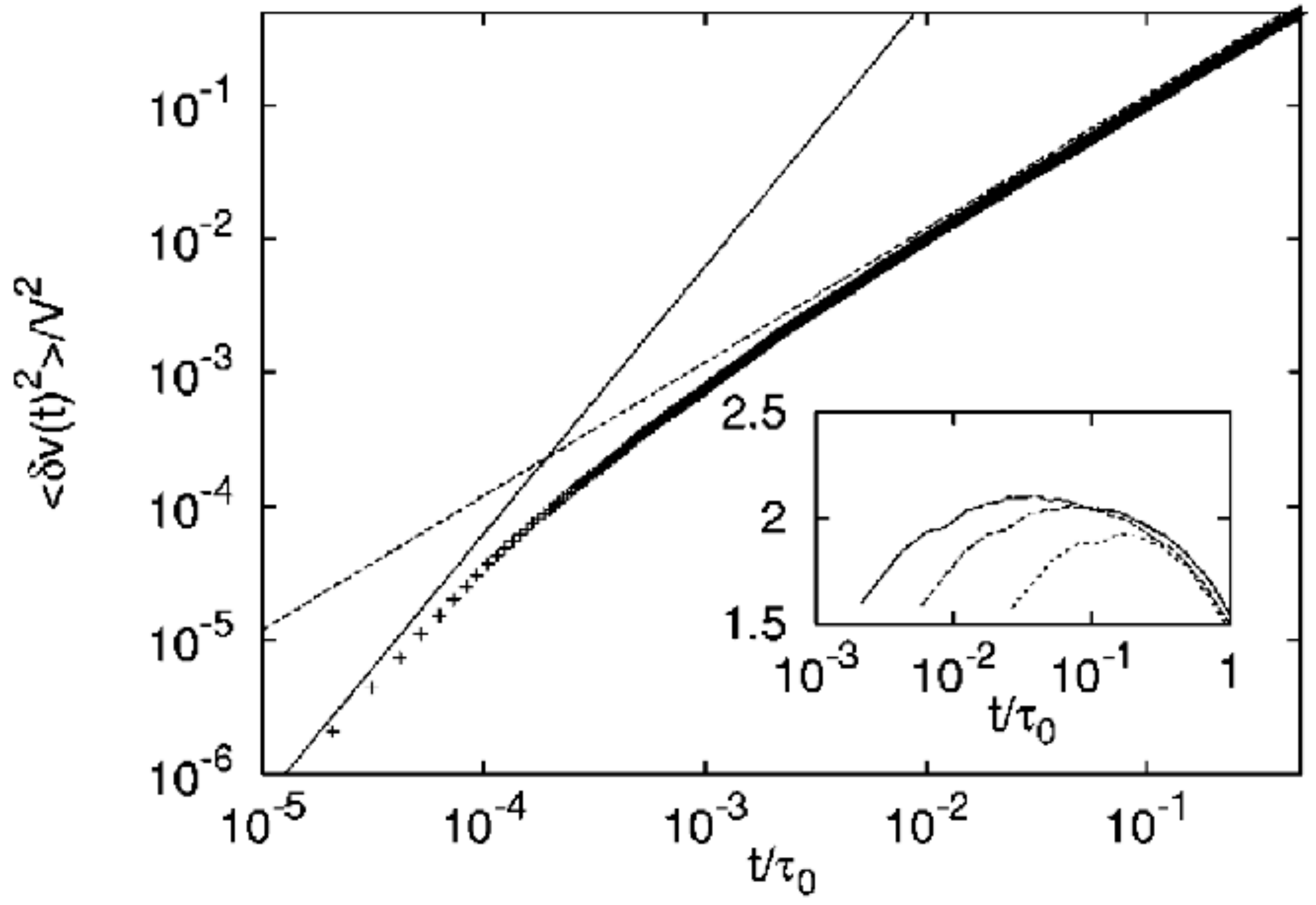


FIG. 2. Second-order Lagrangian structure function $\langle \delta v(t)^2 \rangle$ normalized with large-scale velocity V as a function of time delay t for the simulation at $\mathrm{Re}=10^8$. The continuous line is the ballistic behavior t^2 at short time. The dashed line represents the linear growth (1). Inset: $\langle \delta v(t)^2 \rangle$ compensated with the dimensional prediction εt at $\mathrm{Re}=10^8$ (continuous line), $\mathrm{Re}=2\times10^6$ (dashed line), and $\mathrm{Re}=10^5$ (dotted line).

Setting $f_n=\nu=0$, Eq. (5) becomes a conservative system with two conserved quantities which depend on the value of δ [12]. In statistically stationary conditions, the model shows equipartition of the conserved quantities among the shells, in agreement with the statistical mechanics prediction [17]. For $\delta=1+2^{-2/3}$ the equipartition state leads at small scales to Kolmogorov scaling $\langle |u_n|^2 \rangle \sim k_n^{-2/3}$ with Gaussian statistics.

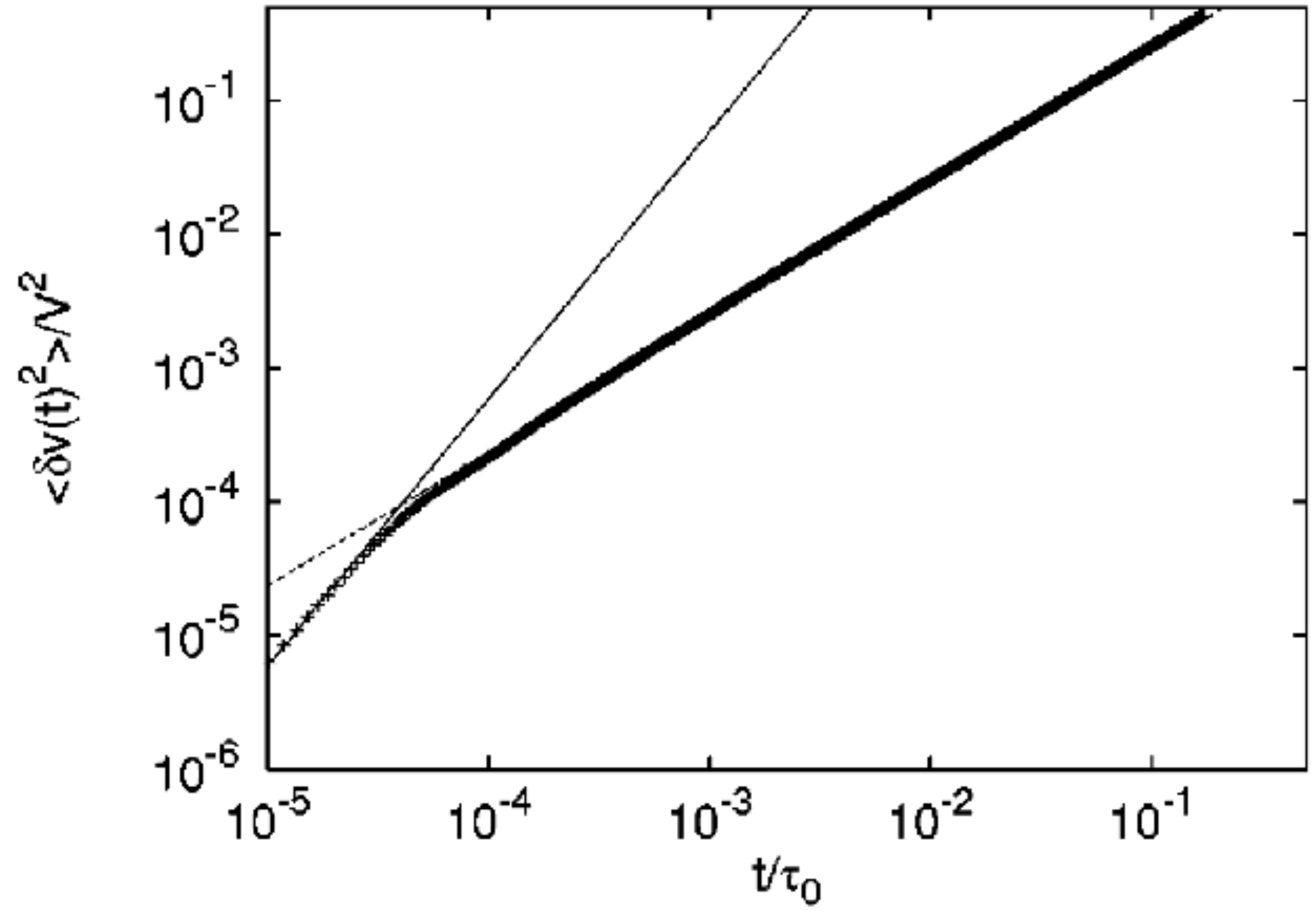


FIG. 5. Second-order Lagrangian structure function $\langle \delta v(t)^2 \rangle$ normalized with large-scale velocity V as a function of time delay t for the equilibrium Gaussian model. The continuous line is the ballistic behavior t^2 at short time. The dashed line represents the linear growth (1).

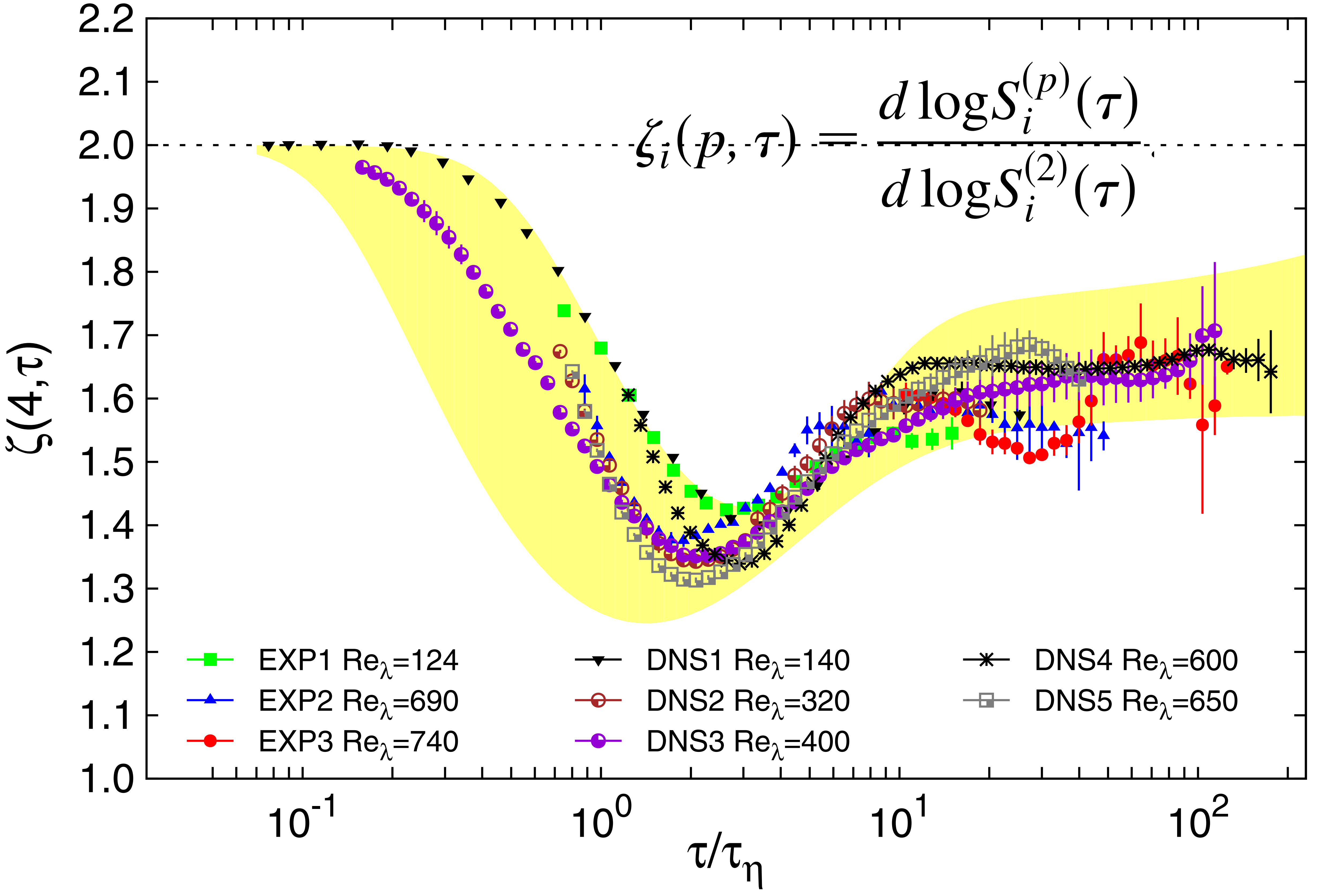
G. Boffetta, F. De Lillo, S Musacchio PRE 66, 066307 (2002)

Test of MF in the Lagrangian frame (DNS+EXP)

PRL **100,** 254504 (2008) PHYSICAL REVIEW LETTERS week ending 27 JUNE 2008

Universal Intermittent Properties of Particle Trajectories in Highly Turbulent Flows

A. Arnèodo,[1] R. Benzi,[2] J. Berg,[3] L. Biferale,[4,*] E. Bodenschatz,[5] A. Busse,[6] E. Calzavarini,[7] B. Castaing,[1] M. Cencini,[8,*] L. Chevillard,[1] R. T. Fisher,[9] R. Grauer,[10] H. Homann,[10] D. Lamb,[9] A. S. Lanotte,[11,*] E. Lévèque,[1] B. Lüthi,[12] J. Mann,[3] N. Mordant,[13] W.-C. Müller,[6] S. Ott,[3] N. T. Ouellette,[14] J.-F. Pinton,[1] S. B. Pope,[15] S. G. Roux,[1] F. Toschi,[16,17,*] H. Xu,[5] and P. K. Yeung[18]



Batchelor parametrization

L. Chevillard *et al.*, Phys. Rev. Lett. **91**, 214502 (2003)

$$\delta_\tau v(h) = V_0 \frac{\tau}{T_L}\left[\left(\frac{\tau}{T_L}\right)^\beta + \left(\frac{\tau_\eta}{T_L}\right)^\beta\right]^{(2h-1)/\beta(1-h)},$$

$$\langle(\delta_\tau v)^p\rangle \sim \int dh P_h(\tau,\tau_\eta)[\delta_\tau v(h)]^p,$$

$$P_h(\tau,\tau_\eta) = Z^{-1}(\tau)\left[\left(\frac{\tau}{T_L}\right)^\beta + \left(\frac{\tau_\eta}{T_L}\right)^\beta\right]^{[3-D(h)]/\beta(1-h)},$$

β a free parameter to model the dissipative cut off

$h_0 = 1/9$

$\gamma = 2/3$ **fitting well longitudinal SF**

$\gamma = 0.5$ **fitting well transverse SF**

$\beta = 4$

$$D(h) = \frac{3(h-h_0)}{\log(\gamma)}\left[\log\left(\frac{3(h_0-h)}{d_0\log(\gamma)}\right) - 1\right] + 3 - d_0$$

Lagrangian SF are measured on single components, i.e. $u_i(t+\tau) - u_i(t)$, so it is unclear whether they are linked to longitudinal or transverse increments
these in principle should scale the same but for not too large Re they seem to scale a bit differently

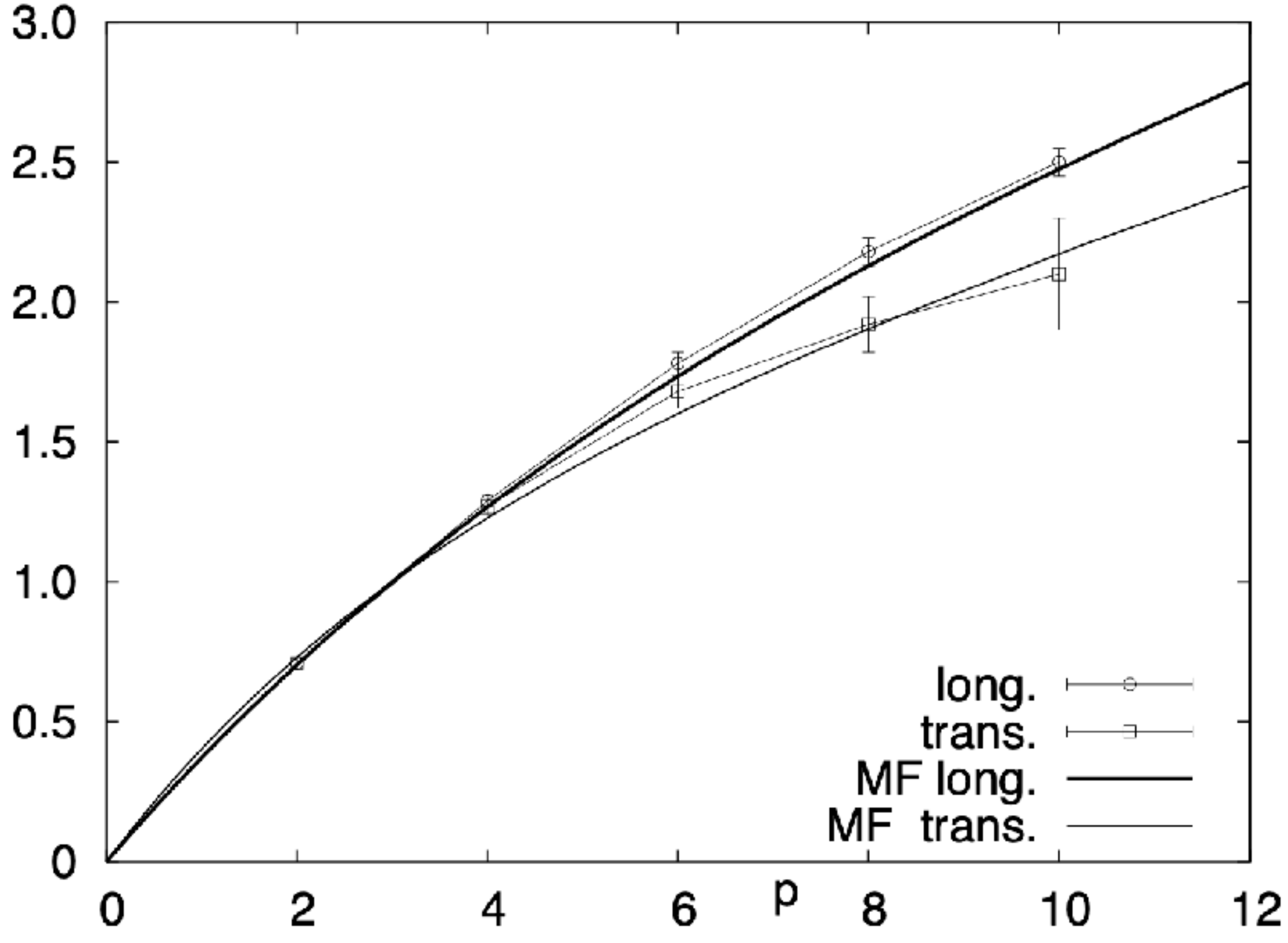


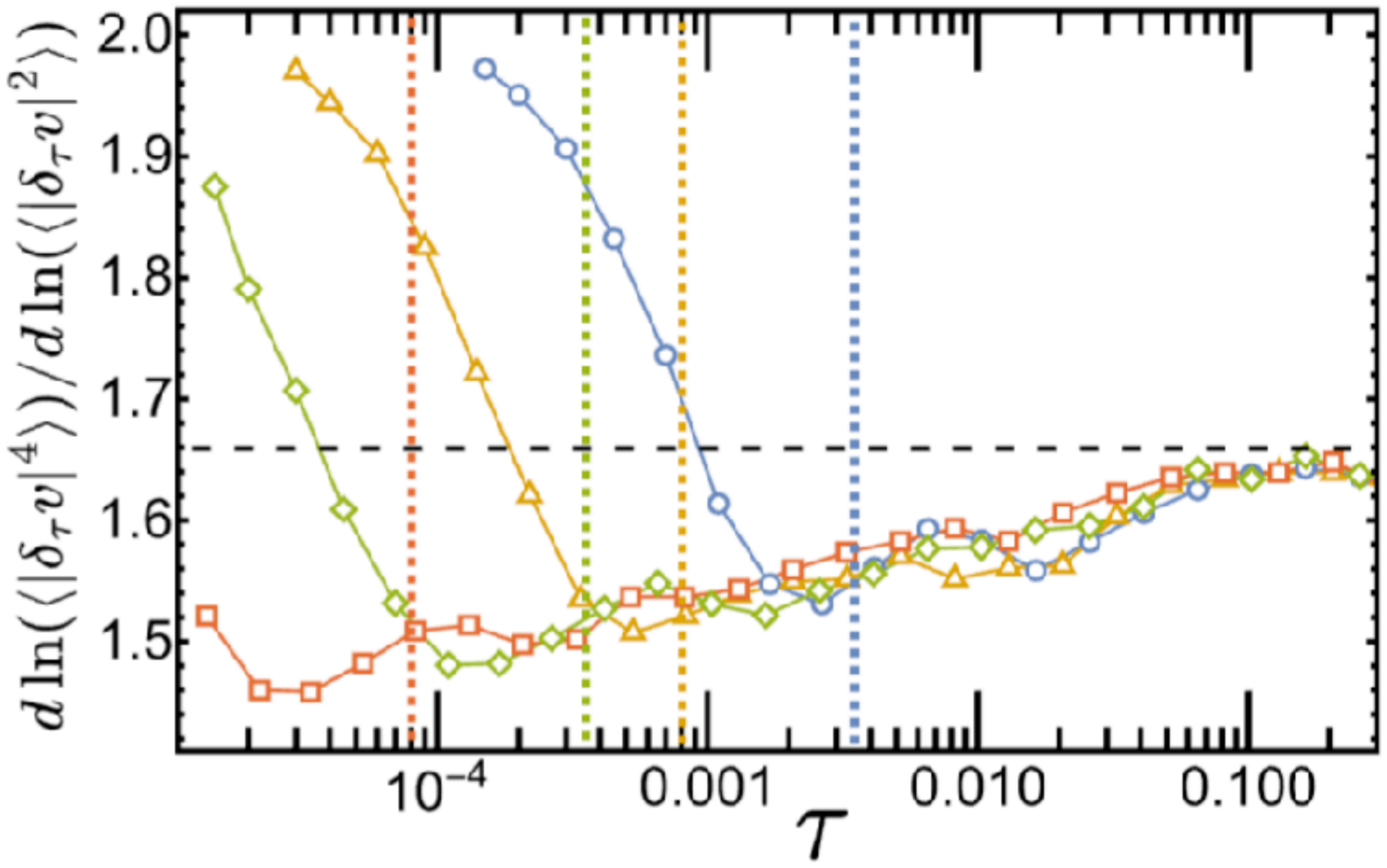
Fig. 6 Comparison between Eulerian scaling exponents for longitudinal, $\zeta_l(p)$, and transverse, $\zeta_{tr}(p)$, Structure Function [11] together with two different multifractal predictions (MF) obtained with two different choices of $D(h)$.

$$S_p^{long}(r) = \langle [(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}}]^p \rangle$$

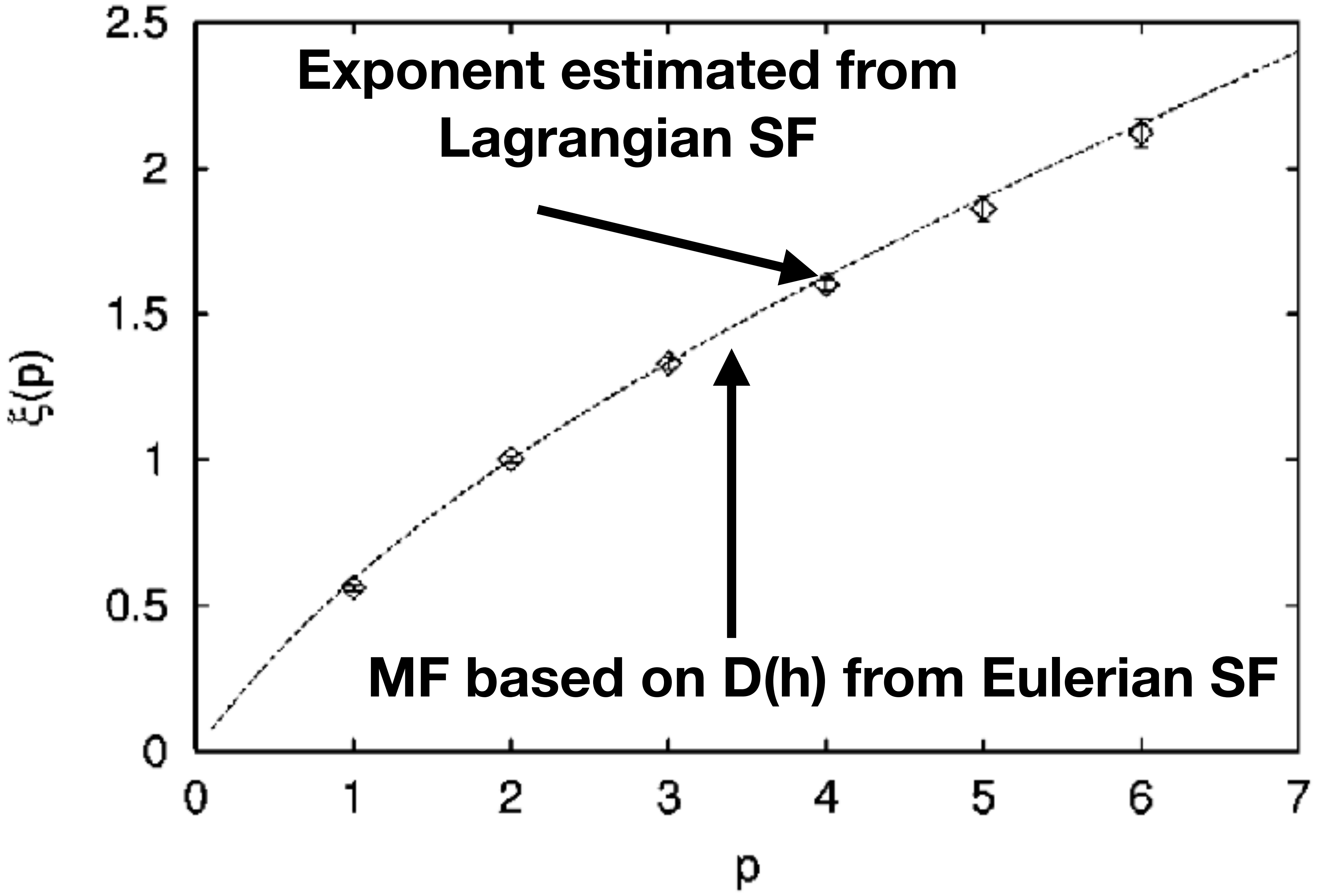
$$S_p^{trans}(r) = \langle [(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}}^{\perp}]^p \rangle$$

Benzi, and Biferale. "Homogeneous and isotropic turbulence: A short survey on recent developments." *Journal of Statistical Physics* 161.6 (2015): 1351-1365.

Test of MF in the Lagrangian frame (Shell model)



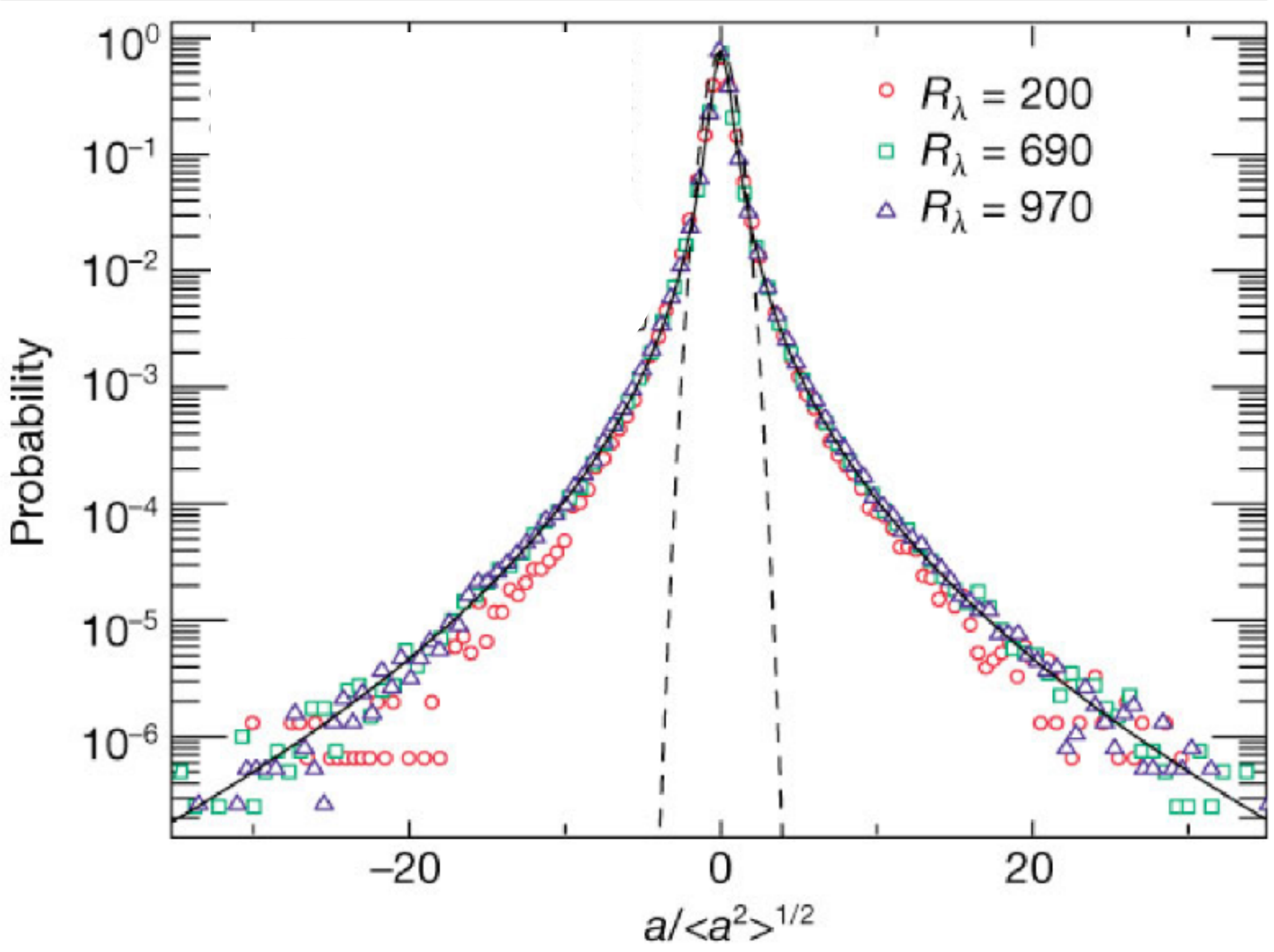
L. Piro, M. Cencini, R. Benzi. PRFluids 10, L092601 (2025)



G. Boffetta, F. De Lillo, S Musacchio PRE 66, 066307 (2002)

$$\xi(p) = \min_h \left[\frac{ph - D(h) + 3}{1 - h} \right]$$

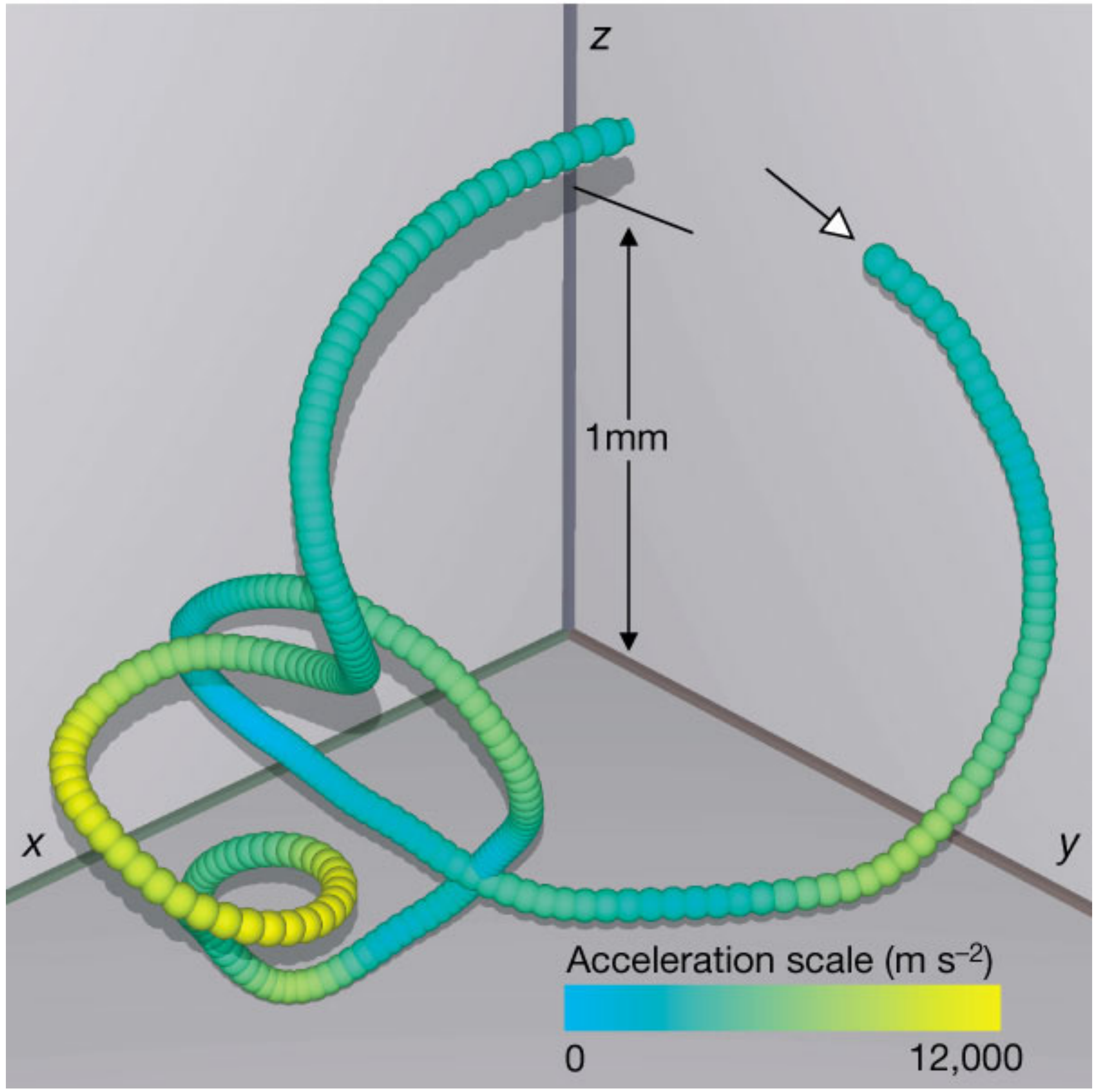
Acceleration statistics



La Porta, Voth, Crawford, Alexander, Bodenschatz, Nature (2001)

$$\dot{\boldsymbol{X}} = \boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t)$$

$$\boldsymbol{A} = \dot{\boldsymbol{v}}(t) = -\boldsymbol{\nabla} p(\boldsymbol{X}(t), t) + \nu \Delta \boldsymbol{u}(\boldsymbol{X}(t), t) + \boldsymbol{F}(\boldsymbol{X}(t), t)$$



Acceleration statistics is extremely intermittent
Extreme events seem to be associated to trapping in vortices

Acceleration statistics

What to expect on dimensional ground (K41)

$$a \propto \frac{\delta_{\tau_\eta} u}{\tau_\eta} \propto \frac{(\delta_\eta u)^2}{\eta} = \frac{(\delta_L u)^2}{L}\left(\frac{\eta}{L}\right)^{-1/3} = \frac{(\delta_L u)^2}{L} Re^{1/4}$$

or equivalently (assuming only ϵ and ν matters)

$$a = a_0 \epsilon^{3/4} \nu^{-1/4} = a_0 \frac{(\delta_L u)^2}{L} Re^{1/4}$$

$$\langle |a|^p \rangle \sim Re^{p/4} \sim Re_\lambda^{p/2}$$

Acceleration statistics

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There are clear deviations from the dimensional prediction can we rationalize theem within the MF model?

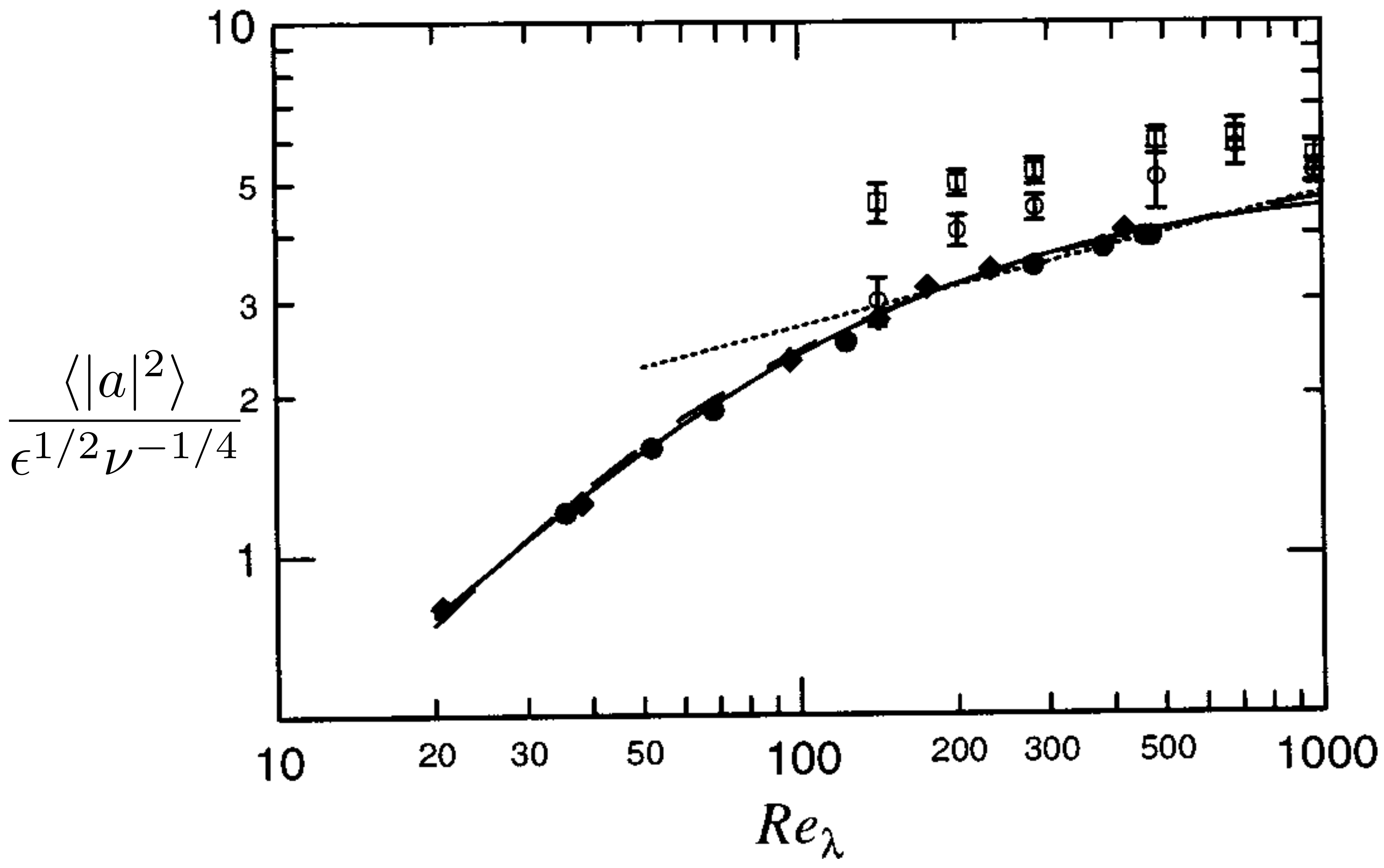

FIG. 1. $a_0 = \frac{1}{3}\langle A_i^2 \rangle / (\langle \epsilon \rangle^{3/2} \nu^{-1/2})$ as a function of Taylor-scale Reynolds number. Symbols are: DNS data, Gotoh and Fukayama (●), Vedula and Yeung (◆); laboratory data with error bars, axial (□), transverse (○). Lines are: $5/(1+110/\mathrm{Re}_\lambda)$ (——), $1.9\,\mathrm{Re}_\lambda^{0.135}/(1+85/\mathrm{Re}_\lambda^{1.135})$ (— — —), $0.85\,\mathrm{Re}_\lambda^{0.25}$ (· · · ·).

B.L. Sawford et al PoF 15, 3478 (2003)

Acceleration statistics: MF prediction

$$a \propto \frac{\delta_{\tau_\eta} u}{\tau_\eta} \propto \frac{(\delta_\eta u)^2}{\eta} = \frac{(\delta_L u)^2}{L}\left(\frac{\eta}{L}\right)^{2h-1} = \frac{(\delta_L u)^2}{L} Re^{\frac{1-2h}{1+h}}$$

$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} \sim Re^{\frac{D(h)-3}{1+h}}$$

$$\boxed{\left(\frac{\eta}{L}\right) \sim Re^{-\frac{1}{1+h}}}$$

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$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} \sim Re^{\frac{D(h)-3}{1+h}} \longleftarrow \boxed{\left(\frac{\eta}{L}\right) \sim Re^{-\frac{1}{1+h}}}$$

$$\langle |a|^p \rangle \propto \left[\frac{(\delta_L u)^2}{L}\right]^{\mathbf{p}} \int dh Re^{\frac{(1-2h)p-3+D(h)}{1+h}}$$

Acceleration statistics: MF prediction

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$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} \sim Re^{\frac{D(h)-3}{1+h}}$$

$$\left(\frac{\eta}{L}\right) \sim Re^{-\frac{1}{1+h}}$$

$$\langle |a|^p \rangle \propto \left[\frac{(\delta_L u)^2}{L}\right]^{\mathbf{p}} \int dh Re^{\frac{(1-2h)p-3+D(h)}{1+h}}$$

$$\langle |a|^p \rangle \propto \left[\frac{(\delta_L u)^2}{L}\right]^{\mathbf{p}} Re^{\alpha_p}$$

$$\alpha_p = \sup_h \left\{ \frac{(1-2h)p - 3 + D(h)}{1+h} \right\}$$

Acceleration statistics: MF prediction

$$a \propto \frac{\delta_{\tau_\eta} u}{\tau_\eta} \propto \frac{(\delta_\eta u)^2}{\eta} = \frac{(\delta_L u)^2}{L}\left(\frac{\eta}{L}\right)^{2h-1} = \frac{(\delta_L u)^2}{L} Re^{\frac{1-2h}{1+h}}$$

$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} \sim Re^{\frac{D(h)-3}{1+h}}$$

$$\boxed{\left(\frac{\eta}{L}\right) \sim Re^{-\frac{1}{1+h}}}$$

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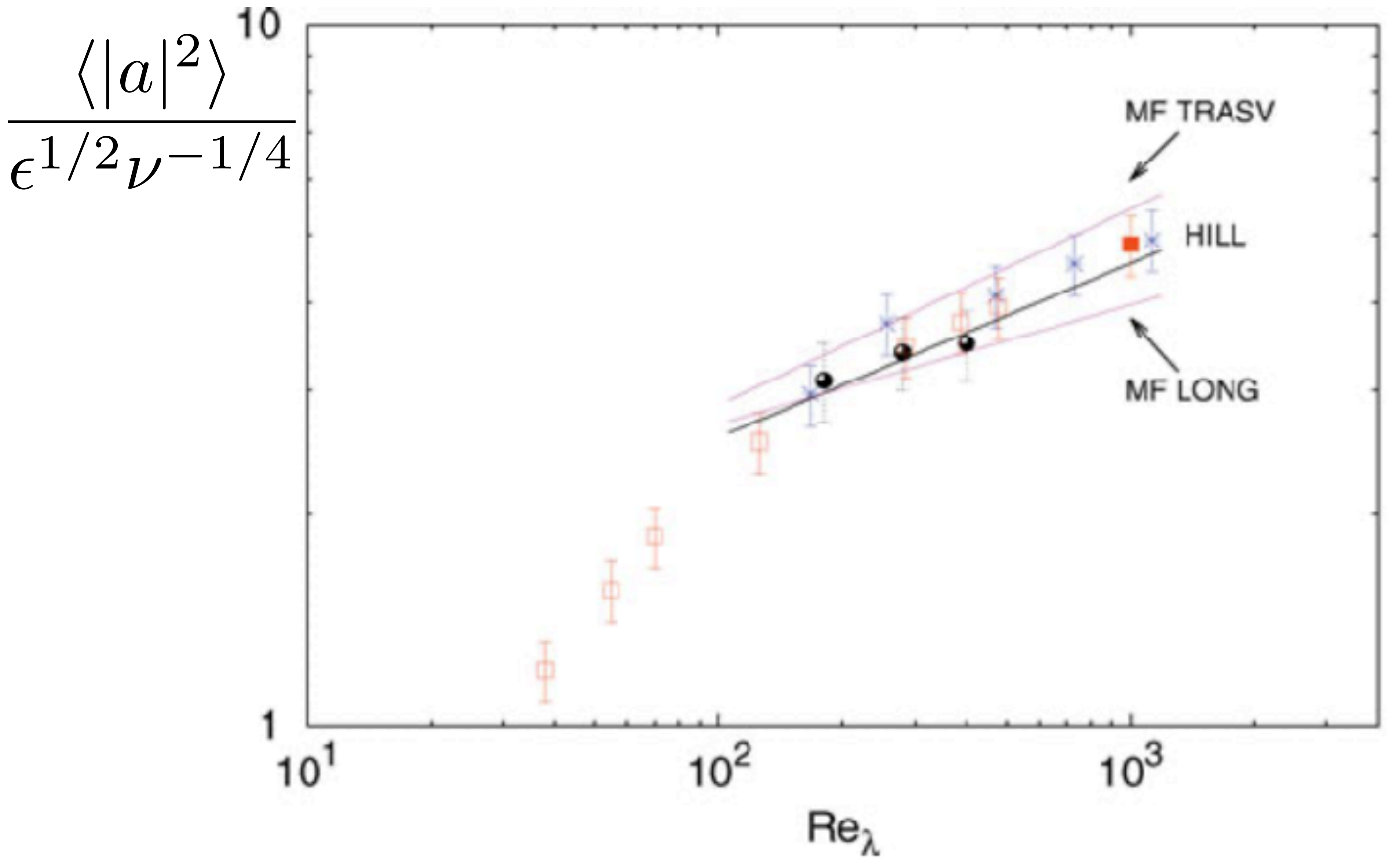
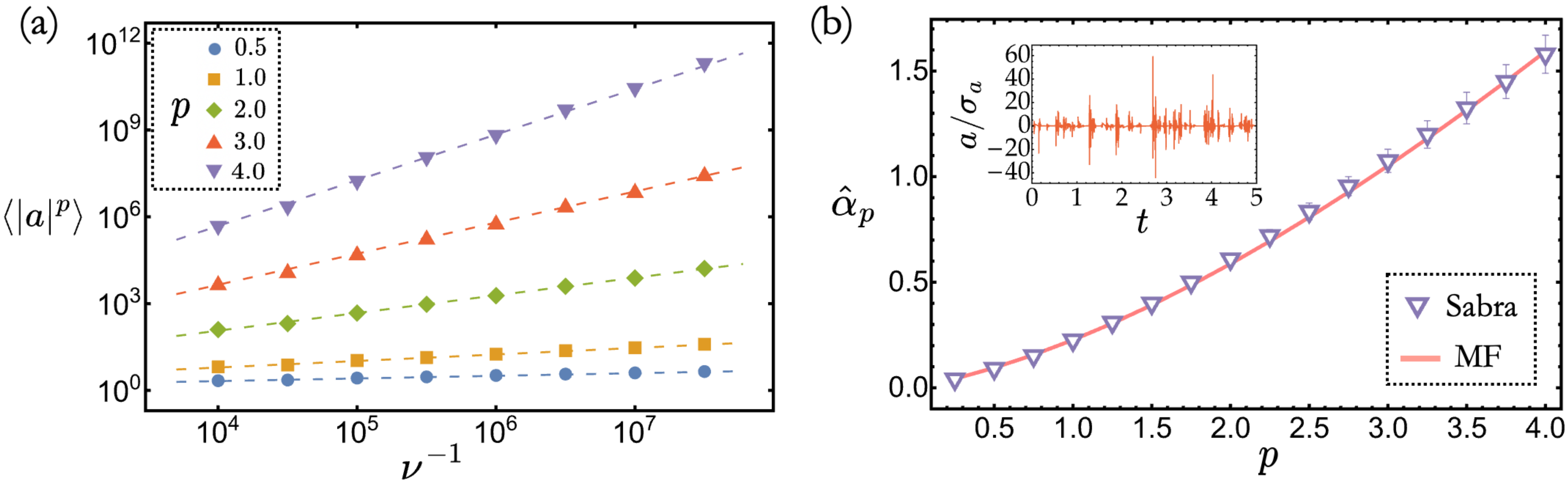


Figure 4. Collection of different numerical data of the scaling of normalised root-mean-square acceleration as a function of the Taylor-scale based Reynolds number Re_λ. Two lines correspond to the multi-fractal prediction using the bridge relation for transverse increments (MF TRASV) leading to $\gamma = 0.17$, or the bridge relation for longitudinal increments (MF LONG) leading to $\gamma = 0.28$ (see [10] for details). These lines can be shifted up or down arbitrarily, being the multi-fractal prediction valid scaling-wise and not for the prefactors. A third line is a fit proposed by Hill in [34], as a superposition of two power laws of exponents $\gamma_1 = 0.25$ and $\gamma_2 = 0.11$. Data are taken from Refs. [11,30,31,35–37]. Error bars are estimated considering a typical 10% uncertainty in the energy dissipation rate.

Using She-Leveque model

A.S. Lanotte et al. JoT 14., 34-48 (2013)

Results on the shell model



$$v(t) \equiv \sum_{n=1}^{N} \Re\{u_n\}\,, \quad a(t) \equiv \dot{v}(t) \equiv \sum_{n=1}^{N} \Re\{\dot{u}_n\}$$

$$\dot{u}_n = ik_n(u_{n+2}u^*_{n+1} - \frac{1}{4}u_{n+1}u^*_{n-1} + \frac{1}{8}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n$$

L. Piro, M. Cencini, R. Benzi. PRFluids 10, L092601 (2025)

Acceleration PDF

$U = \delta_L u$ now we consider that U is a fluctuating quantity and assume Gaussian statistics $P(U) = \frac{1}{\sqrt{2\pi\sigma_U^2}} e^{-\frac{U^2}{2\sigma_U^2}}$

intermediate dissipative range

$$a \propto \frac{(\delta_\eta u)^2}{\eta} \sim \frac{U^2}{L}\left(\frac{\eta}{L}\right)^{2h-1} \qquad \left(\frac{\eta}{L}\right) \approx \left(\frac{\nu}{LU}\right)^{\frac{1}{(1+h)}}$$

$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} = \left(\frac{\nu}{LU}\right)^{\frac{3-D(h)}{1+h}}$$

$$a = a(h,U) = U^{\frac{3}{1+h}} \nu^{\frac{2h-1}{1+h}} L^{-\frac{3h}{1+h}}$$

$$U = a^{\frac{1+h}{3}} \nu^{\frac{1-2h}{3}} L^h$$

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$$P(h) \sim \left(\frac{\eta}{L}\right)^{3-D(h)} = \left(\frac{\nu}{LU}\right)^{\frac{3-D(h)}{1+h}}$$

$$a = a(h,U) = U^{\frac{3}{1+h}} \nu^{\frac{2h-1}{1+h}} L^{-\frac{3h}{1+h}} \quad \Rightarrow \quad U = a^{\frac{1+h}{3}} \nu^{\frac{1-2h}{3}} L^h$$

$$P(a) = Z \int dh P(h) P[U(a,h)] dU/da$$

Acceleration PDF

$U = \delta_L u$ now we consider that U is a fluctuating quantity and assume Gaussian statistics $P(U) = \frac{1}{\sqrt{2\pi\sigma_U^2}} e^{-\frac{U^2}{2\sigma_U^2}}$

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$$P(a) = Z \int dh P(h) P[U(a,h)] dU/da$$

$$\frac{dU}{da} = a^{\frac{h-2}{3}} \nu^{\frac{1-2h}{3}} L^h$$

Acceleration PDF

$U = \delta_L u$ now we consider that U is a fluctuating quantity and assume Gaussian statistics $P(U) = \frac{1}{\sqrt{2\pi\sigma_U^2}} e^{-\frac{U^2}{2\sigma_U^2}}$

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$$P(a) = Z \int dh P(h) P[U(a,h)] dU/da$$

$$\frac{dU}{da} = a^{\frac{h-2}{3}} \nu^{\frac{1-2h}{3}} L^h$$

$$P(a) = Z \int dh a^{\frac{h-5-D(h)}{3}} \nu^{\frac{7-2h-2D(h)}{3}} L^{D(h)+h-3} \sigma_v^{-1} \exp\left(-\frac{a^{\frac{2(1+h)}{3}} \nu^{\frac{2(1-2h)}{3}} L^{2h}}{2\sigma_v^2}\right)$$

VOLUME 93, NUMBER 6 PHYSICAL REVIEW LETTERS week ending 6 AUGUST 2004

Multifractal Statistics of Lagrangian Velocity and Acceleration in Turbulence

L. Biferale,[1] G. Boffetta,[2,6] A. Celani,[3] B. J. Devenish,[1,7] A. Lanotte,[4] and F. Toschi[5]

Acceleration PDF

$$P(a) = Z \int dh\, a^{\frac{h-5-D(h)}{3}} \nu^{\frac{7-2h-2D(h)}{3}} L^{D(h)+h-3} \sigma_v^{-1} \exp\left(- \frac{a^{\frac{2(1+h)}{3}} \nu^{\frac{2(1-2h)}{3}} L^{2h}}{2\sigma_v^2} \right)$$

$$\langle |a|^p \rangle \sim Re^{\alpha_p} \qquad \alpha_p = \sup_h \left\{ \frac{(1-2h)p - 3 + D(h)}{1+h} \right\} \qquad \sigma_a = \langle a^2 \rangle^{1/2} \sim Re^{\alpha_2/2} \qquad Re = \frac{\sigma_v L}{\nu}$$

$$\tilde{a} = \frac{a}{\sigma_a}$$

$$P(\tilde{a}) = Z \int \mathrm{d}h\, \tilde{a}^{\frac{h-5+D(h)}{3}} Re^{\alpha_2 \frac{5-h-D(h)}{6} + \frac{2h+2D(h)-7}{3}} \exp\left[- \frac{\tilde{a}^{\frac{2(1+h)}{3}} Re^{\frac{2(1-2h)}{3} + \frac{2\alpha_2(1+h)}{3}}}{2} \right]$$

Acceleration PDF

DNS

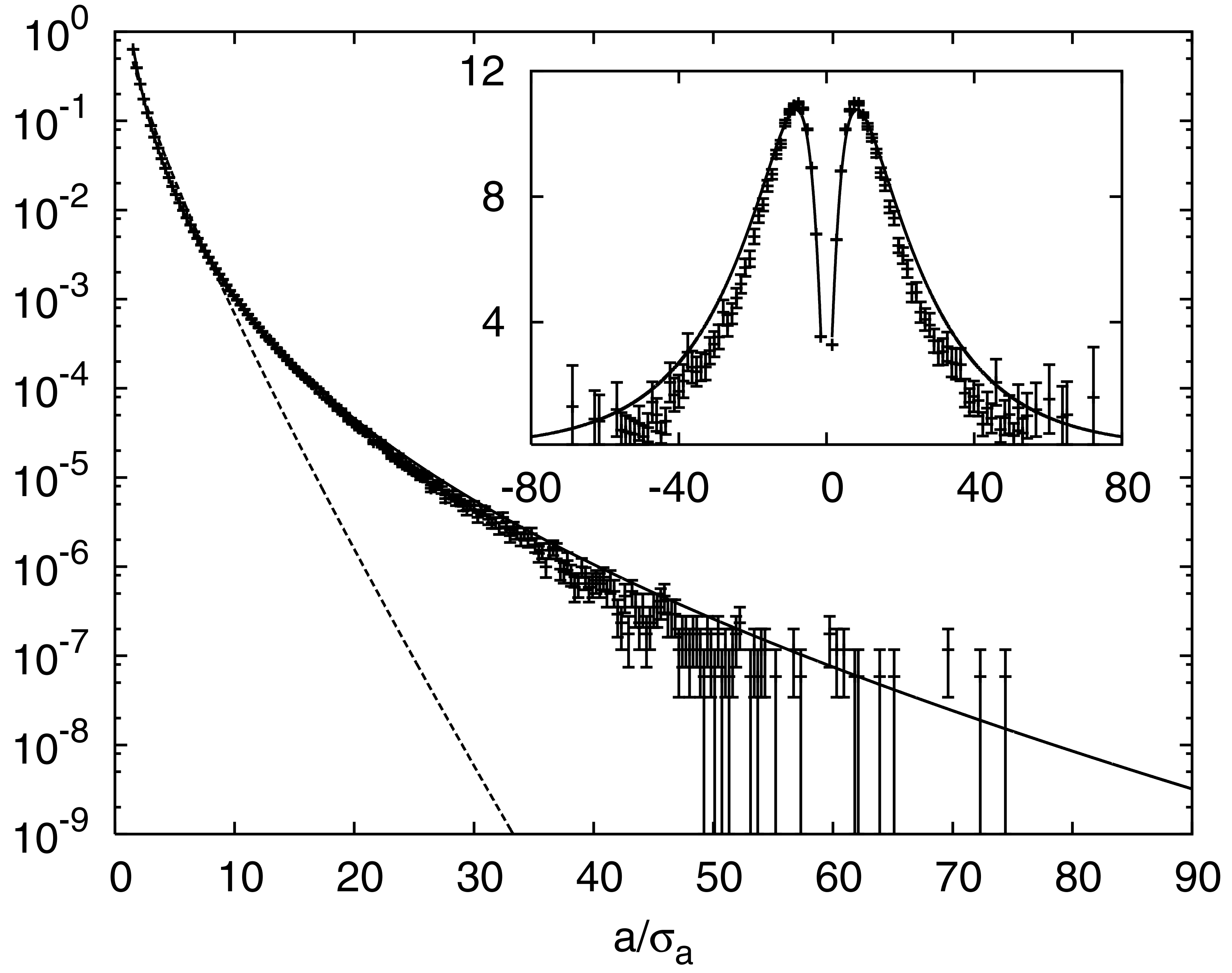


FIG. 2. Log-linear plot of the acceleration PDF. The crosses are the DNS data, the solid line is the multifractal prediction, and the dashed line is the K41 prediction. The DNS statistics were calculated along the trajectories of 2.0×10^6 particles amounting to 1.06×10^{10} events in total. The statistical uncertainty in the PDF was quantified by assuming that fluctuations grow like the square root of the number of events. Inset: $\tilde{a}^4 \mathcal{P}(\tilde{a})$ for the DNS data (crosses) and the multifractal prediction.

L. Biferale et al PRL 93, 064502 (2004)

Shell Model

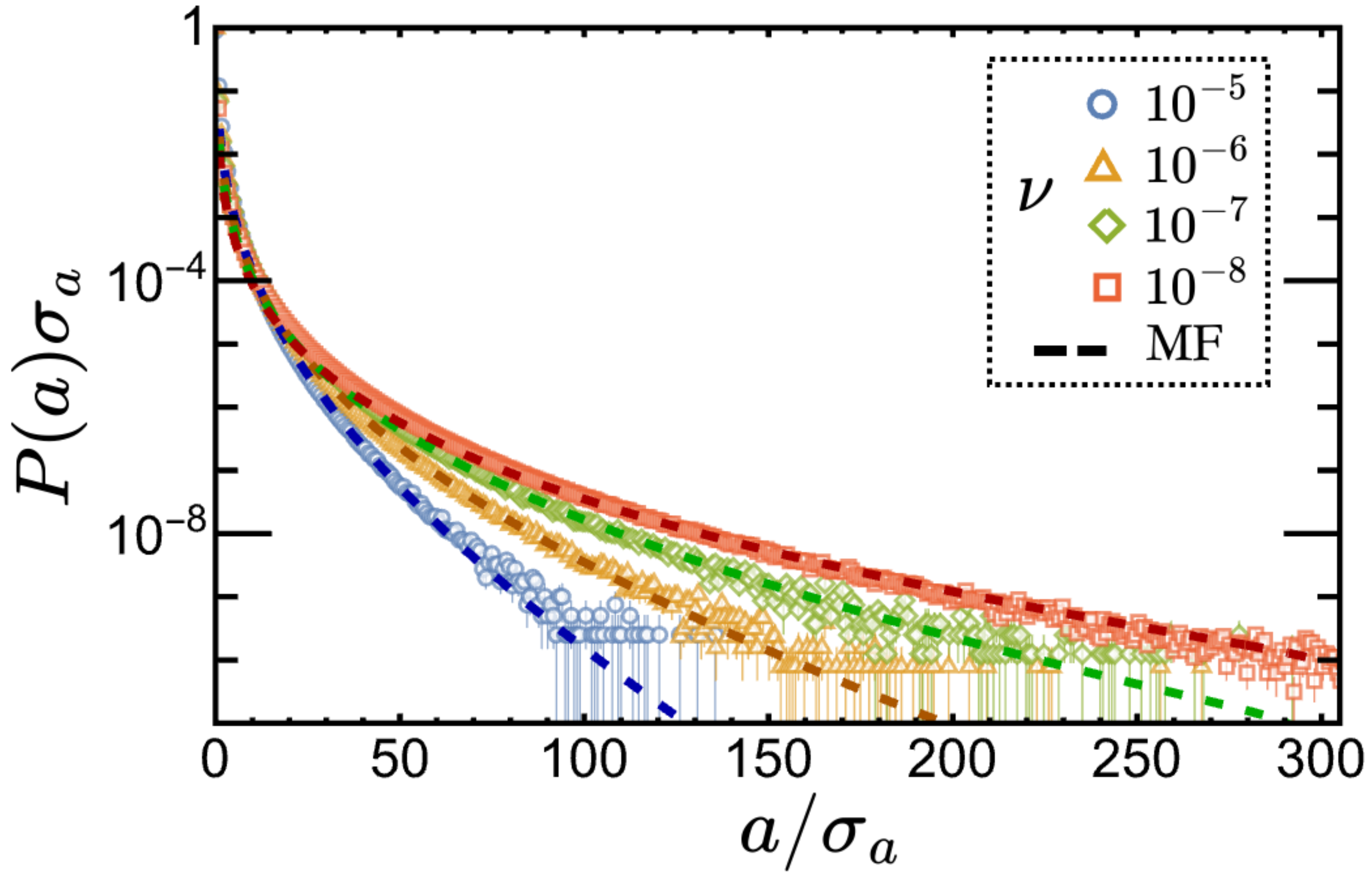


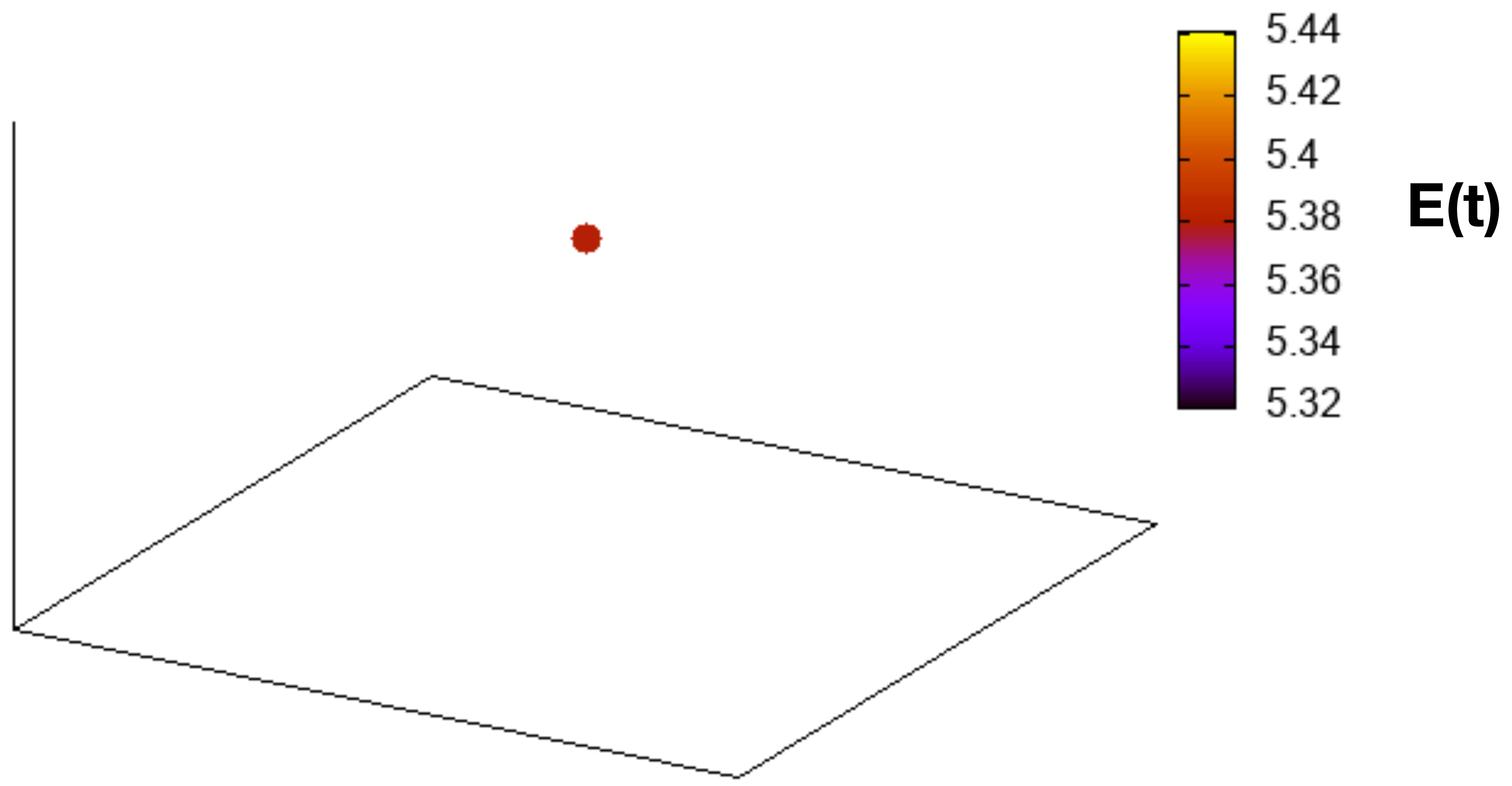
FIG. 3. Log-linear plot of the acceleration PDFs normalized with the standard deviation σ_a of the acceleration for four different values of the viscosity $\nu = \{10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}\}$, as indicated by the legend. Symbols represent the probability distributions obtained from our numerical simulations, while the corresponding dashed lines of matching color indicate the multifractal prediction.

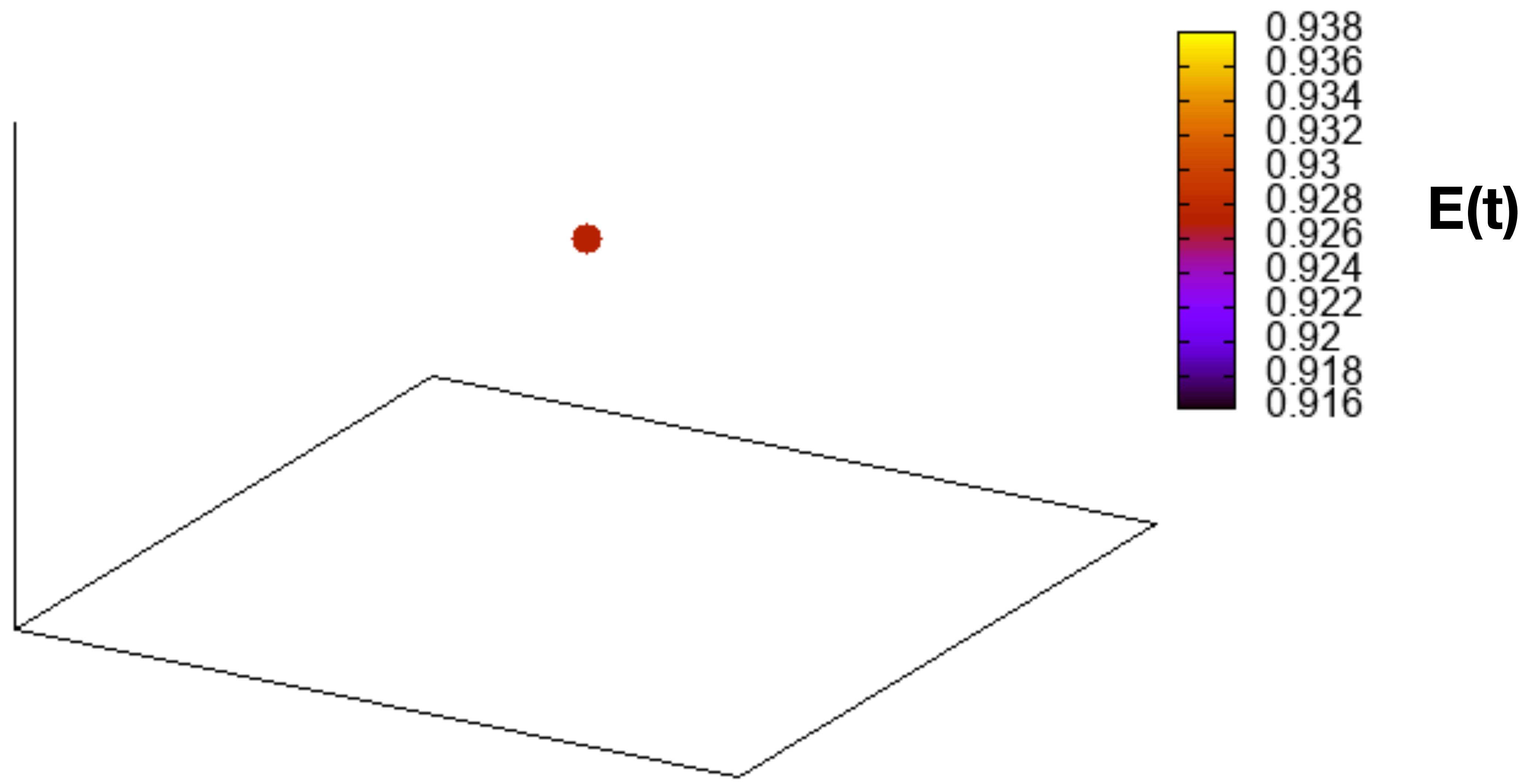
L. Piro et al. PRFluids 10, L092601 (2025)

NB: even though acceleration "may seem" a dissipative range quantity the good agreement with the MF prediction shows that its statistics is inherited from the inertial range physics

Turbulence & Irreversibility

An inviscid-equation symmetry—in this case, time-reversal invariance—remains broken even as the symmetry-breaking viscosity becomes vanishingly small.[4] A trained eye viewing a movie of steady turbulence run backwards can tell that something is indeed wrong! G. Falkovich & K. Sreenivasan Physics Today 2006





Which is forward/backward?

Turbulence & Irreversibility

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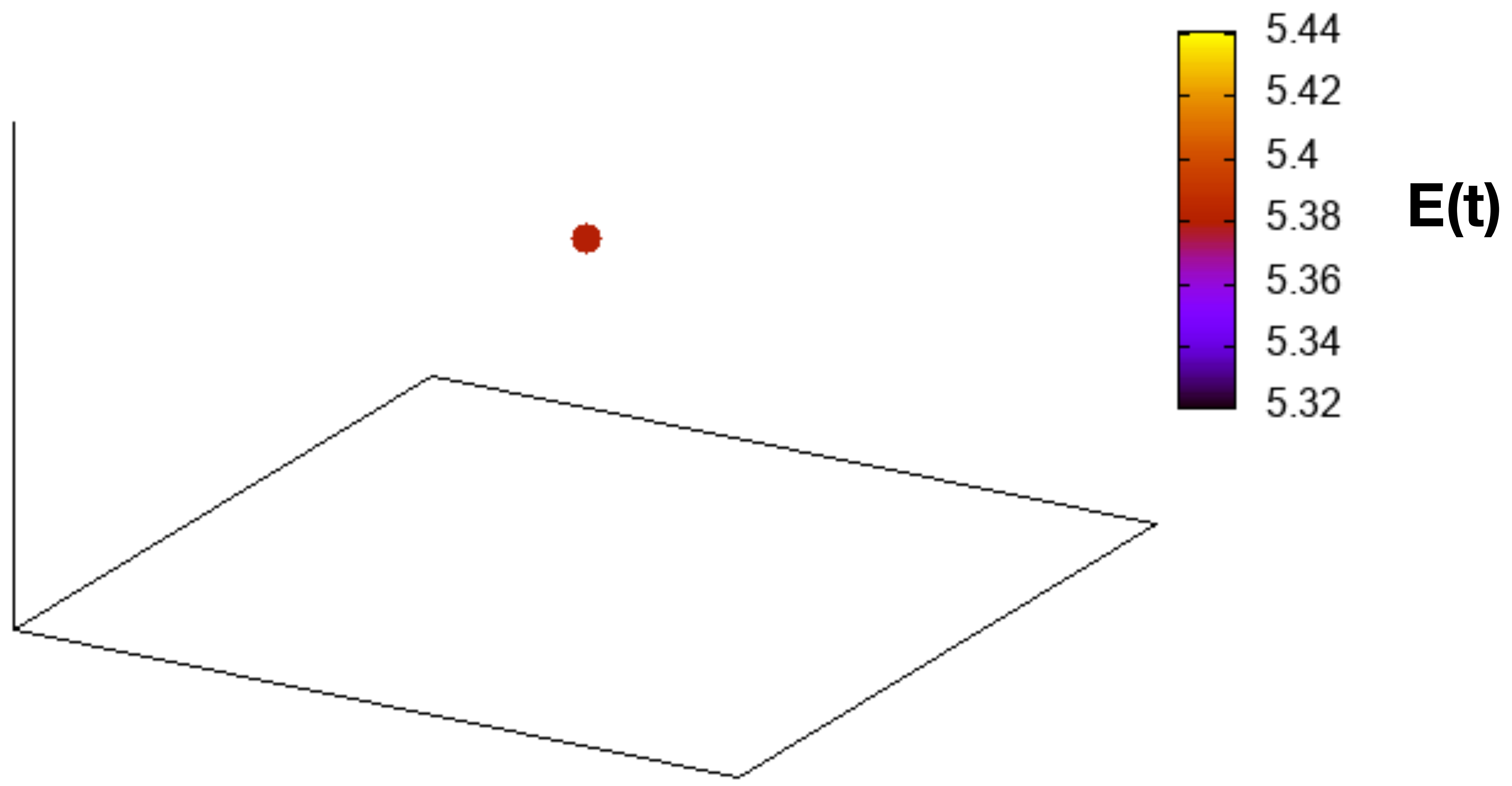

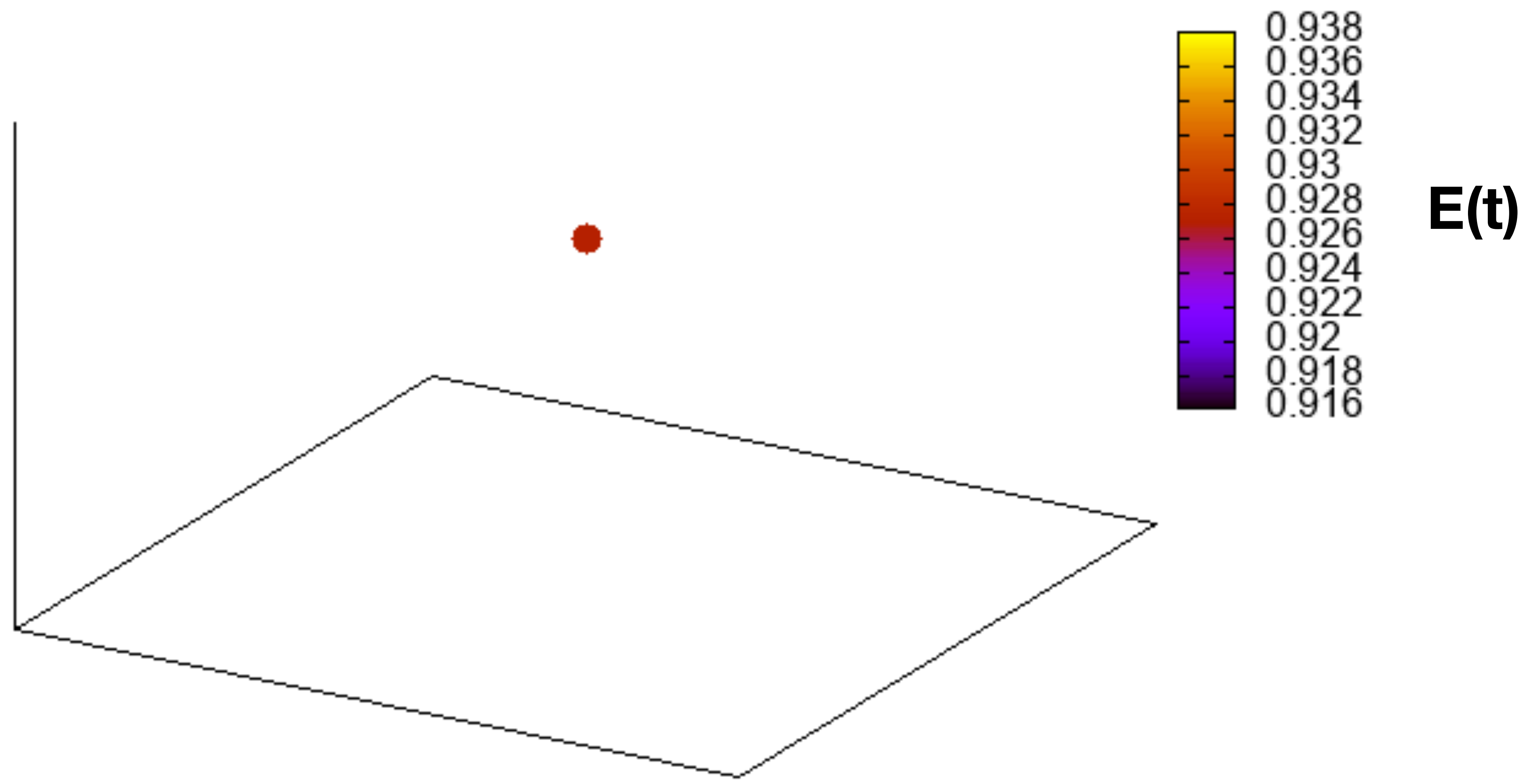

Which is forward/backward?

Take home messages

Anomalous scaling and intermittency characterize the statistics of turbulent velocity field both in the spatial and temporal domain, for the latter the natural framework is the Lagrangian one

The MF model, though being a “phenomenological theory”, is able to rationalize both the Eulerian and Lagrangian points of view

Temporal intermittency is somehow stronger than spatial intermittency and this can also be rationalized in terms of MF model and in particular with the intermediate dissipative range picture

Shell models offer a laboratory to test many ideas allowing to reach very high Reynolds number

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Lectures on turbulence

Roberto Benzi [a], Federico Toschi [b,c,*]

[a] Department of Physics and INFN, University of Rome Tor Vergata, I-00133 Rome, Italy
[b] Department of Applied Physics and Science Education, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands
[c] CNR-IAC, I-00185 Rome, Italy

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Turbulence & Irreversibility

Euler equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p$$

Equilibrium physics:
time reversibility t→-t u→-u

Navier-Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{F}$$

Non-equilibrium and time-reversibility breaking also in the limit $\nu \to 0$

hallmark of time reversal breaking : “4/5 law” (Kolmogorov 1941)

$$S_3(r) = \langle (\delta_{\|} u(r))^3 \rangle = -\frac{4}{5}\epsilon r + 6\nu \partial_r S_2(r) + \dots$$

Looking at the energy cascade (S_3 relates to the energy flux)
we can recognize the arrow of time

How does irreversibility manifests in the Lagrangian frame, i.e. looking at fluid element trajectories?

Flight–crash events in turbulence

PNAS (2014)

Haitao Xu[a,b], Alain Pumir[a,b,c], Gregory Falkovich[a,d,e], Eberhard Bodenschatz[a,b,f,g,1], Michael Shats[h], Hua Xia[h], Nicolas Francois[h], and Guido Boffetta[a,i]

Lagrangian velocity

$$\dot{\mathbf{x}} = \mathbf{v}(t) = \boldsymbol{u}(\mathbf{x}(t), t)$$

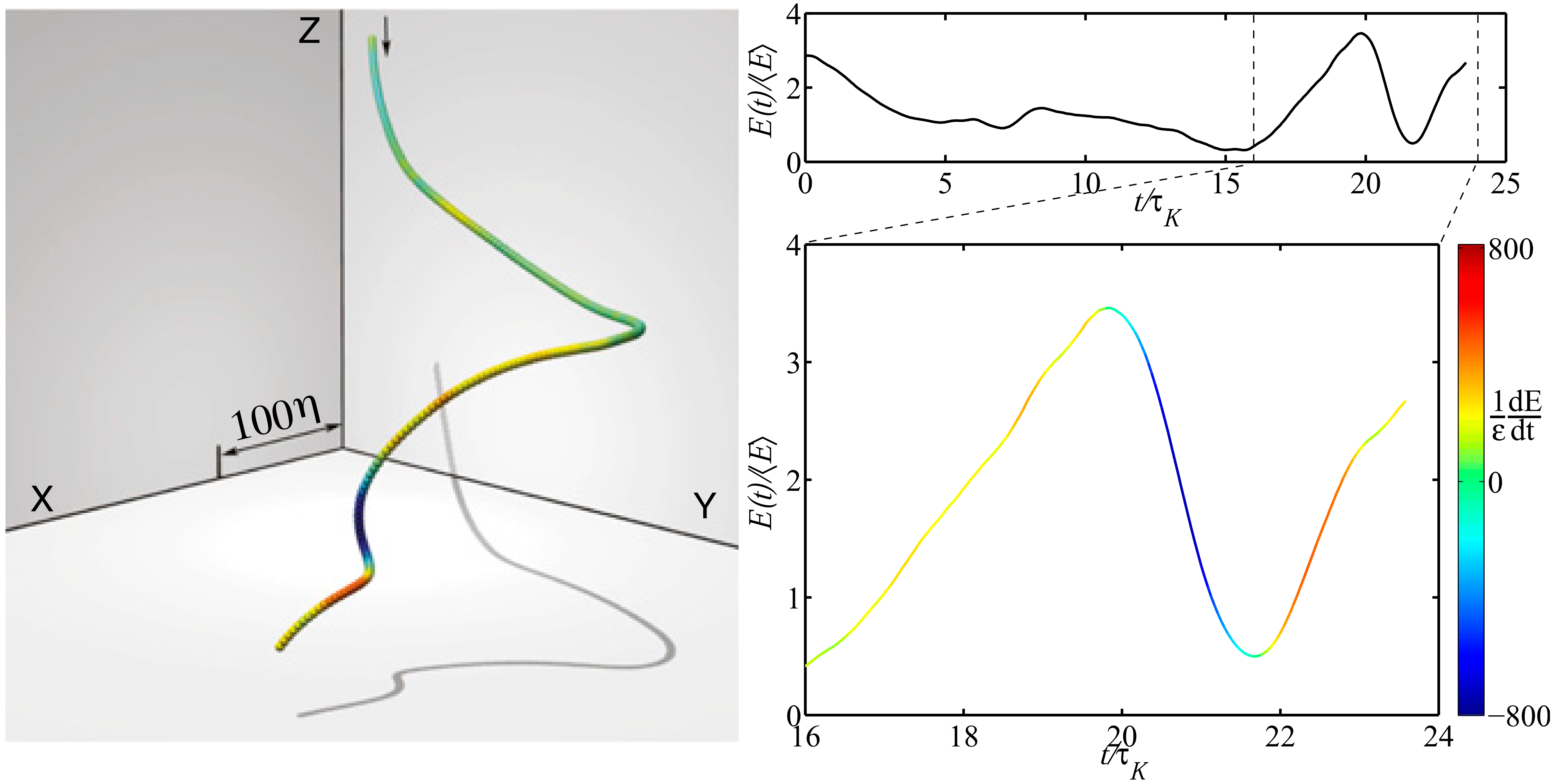
Time increments of Lagrangian kinetic energy are negatively skewed

$$E(t) = \frac{1}{2}|\mathbf{v}^2(t)| \qquad \delta_\tau E = E(t+\tau) - E(t)$$

$$\langle \delta_\tau E \rangle = 0$$ **<- by stationarity**

$$\langle (\delta_\tau E)^3 \rangle \neq 0$$ **<- signature of irreversibility**

Tail of pdf $\delta_\tau E$ is dominated by events in which energy grows more slowly than it decreases



to build up large kinetic energy requires a longer time than to dissipate the same amount

detailed balance breaking

$$P(E \to E + \Delta E) \neq P(E + \Delta E \to E)$$

Lagrangian power statistics

Skewness of $\delta_\tau E$ implies skewness of power

$$\frac{dE}{dt} = p = \boldsymbol{v}\cdot\boldsymbol{a} = \boldsymbol{u}(\mathbf{x}(t),t)\cdot(-\boldsymbol{\nabla}\mathrm{p} + \nu\Delta\boldsymbol{u} + \boldsymbol{F})$$

<p>=0 by stationarity

Irreversibility grows with Re

$$Ir = -\frac{\langle p^3\rangle}{\epsilon^3} \propto Re_\lambda^2$$

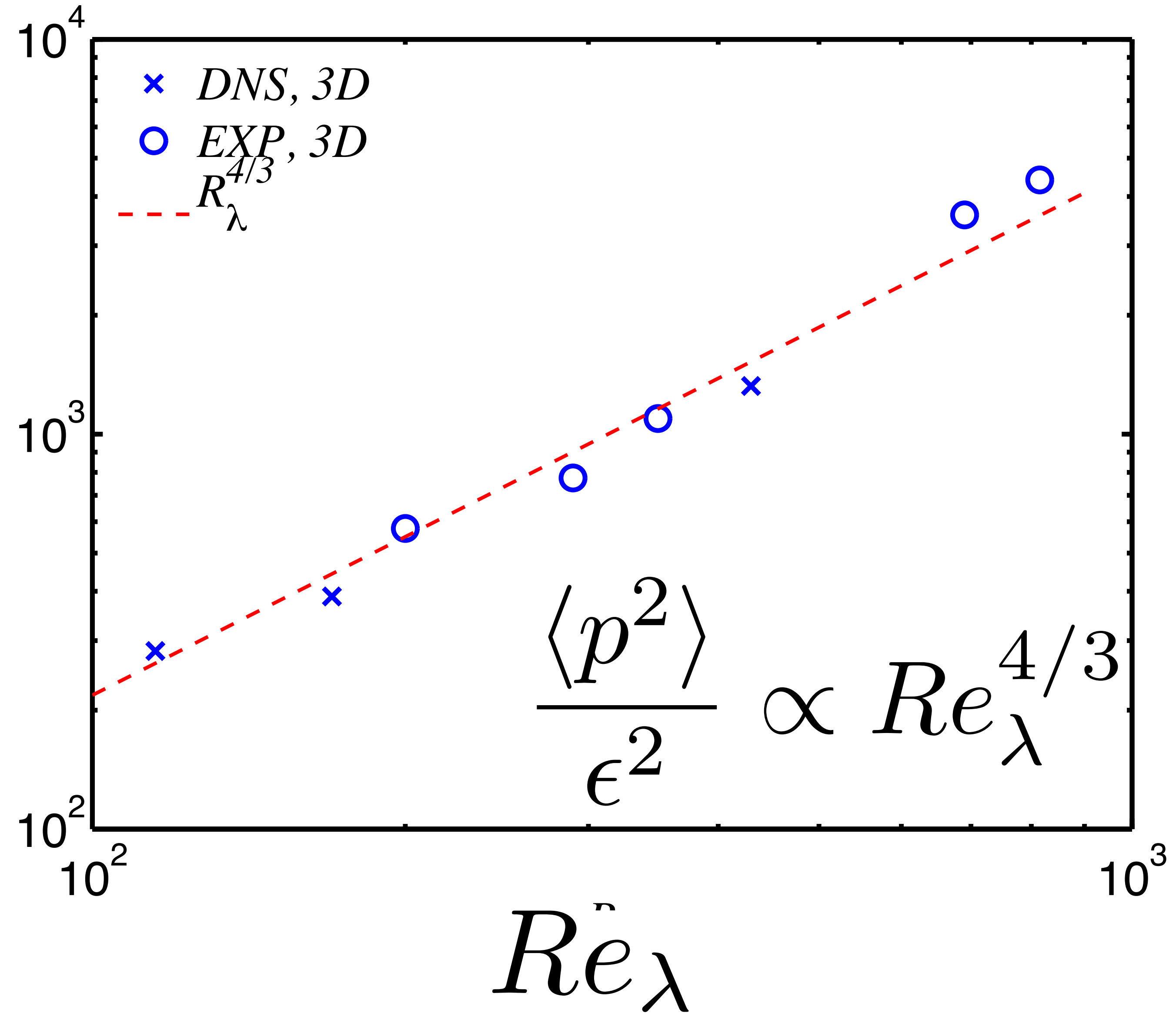
$$Sk = \frac{\langle p^3\rangle}{\langle p^2\rangle^{3/2}} \approx const < 0$$

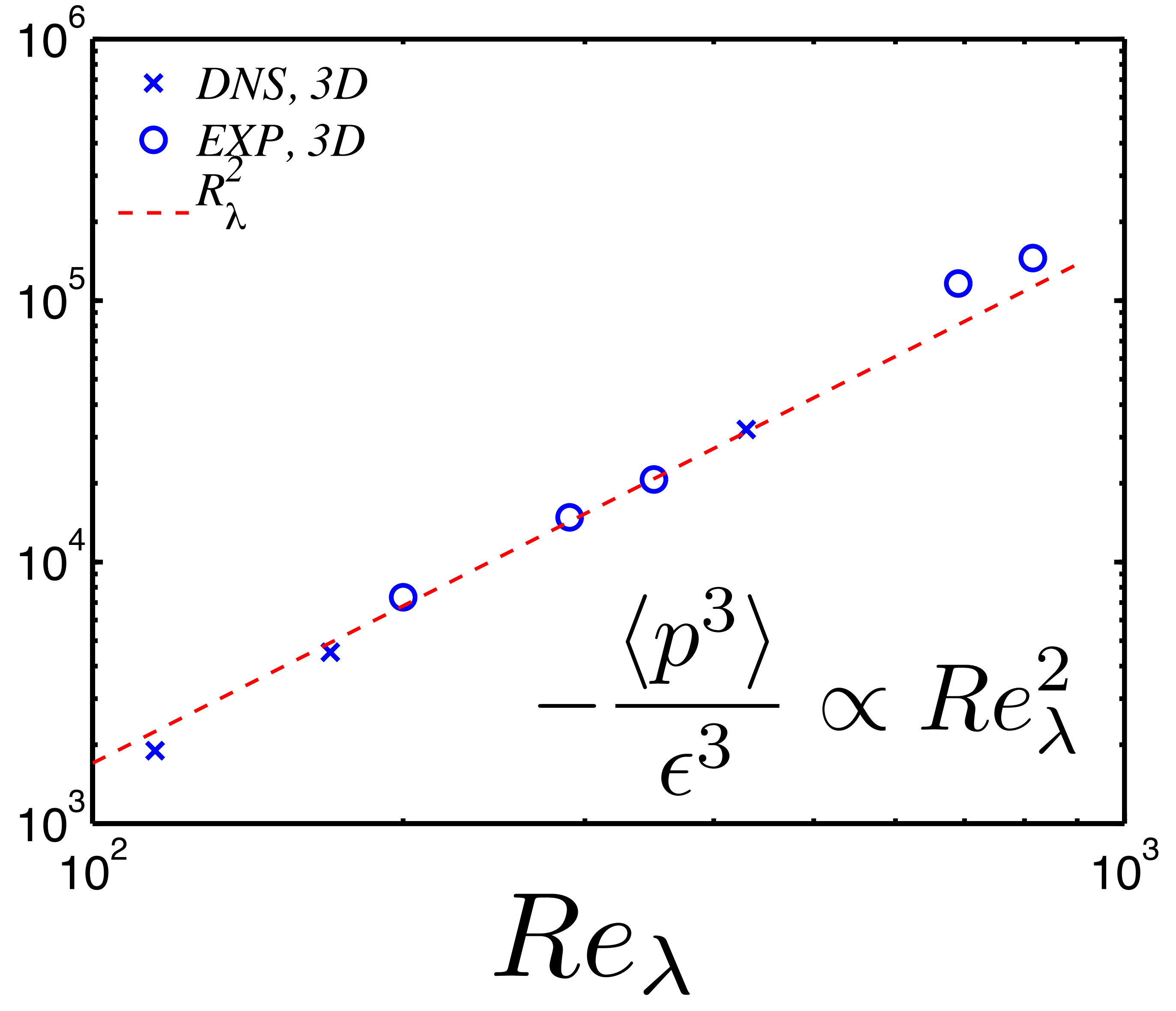
Notice the scaling behavior is anomalous

$$a \approx \frac{u_\eta}{\tau_\eta} \approx \frac{U}{T}Re_\lambda^{1/2} \quad v \approx U \quad \epsilon \approx \frac{U^2}{T} = \frac{U^3}{L}$$

K41➡ $p = \mathbf{v}\cdot\mathbf{a} \approx \epsilon Re_\lambda^{1/2}$

hallmark of intermittency





Flight–crash events in turbulence

PNAS (2014)

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Rationale for the observed scaling

Multifractal (MC, Biferale, Boffetta, De Pietro 2017)

$$\tau \sim \frac{r}{\delta u(r)} \qquad a \sim \nu^{\frac{2h-1}{1+h}} u_L^{\frac{3}{1+h}} L^{\frac{-3h}{1+h}}$$

$$p \sim aU \qquad P(U)$$ Gaussian

$$\langle |a|^q \rangle \sim \langle |p|^q \rangle \sim Re_\lambda^{\alpha(q)} \qquad \alpha(q) = \sup_h \left\{ 2\frac{(1-2h)q - 3 + D(h)}{1+h} \right\}$$

prediction

$$\alpha(2) = 1.17 \qquad \alpha(3) = 2.1$$

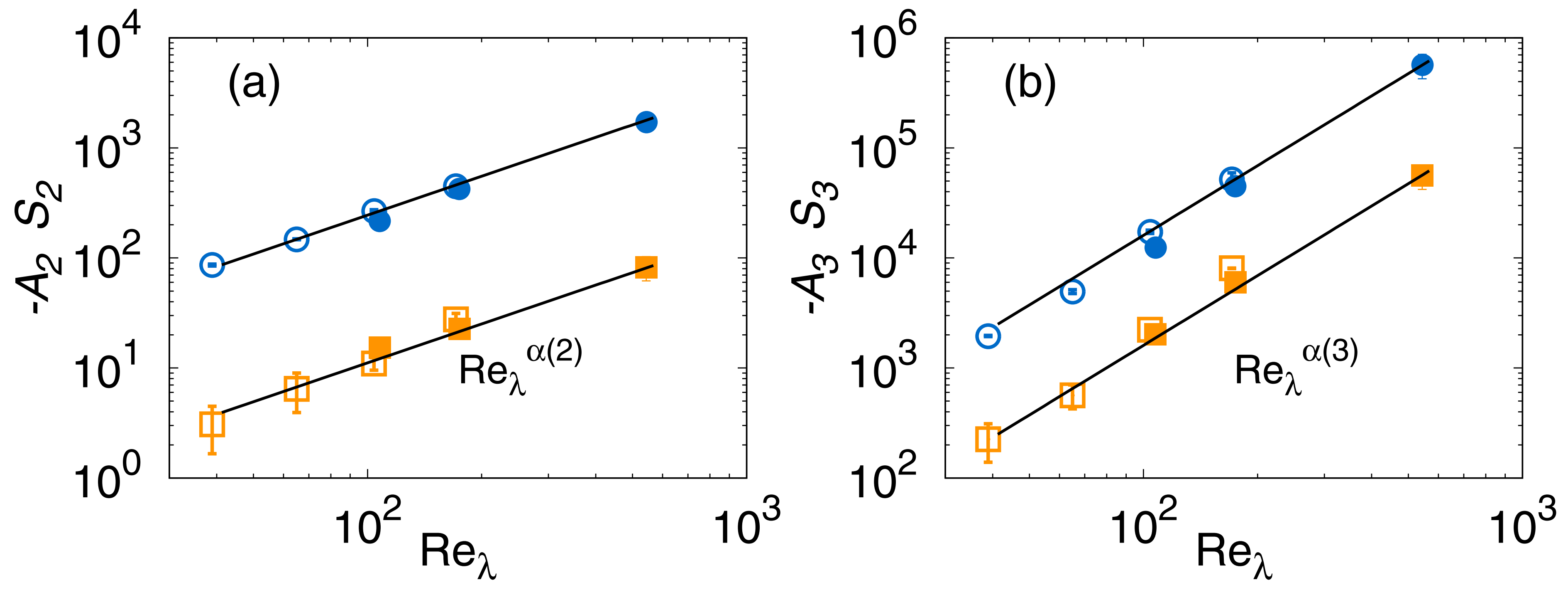
Lagrangian power statistics in NS

Symmetric moments

$\mathcal{S}_q = \langle |p|^q \rangle / \epsilon^q$

Asymmetric moments

$\mathcal{A}_q = \langle p|p|^{q-1} \rangle / \epsilon^q$



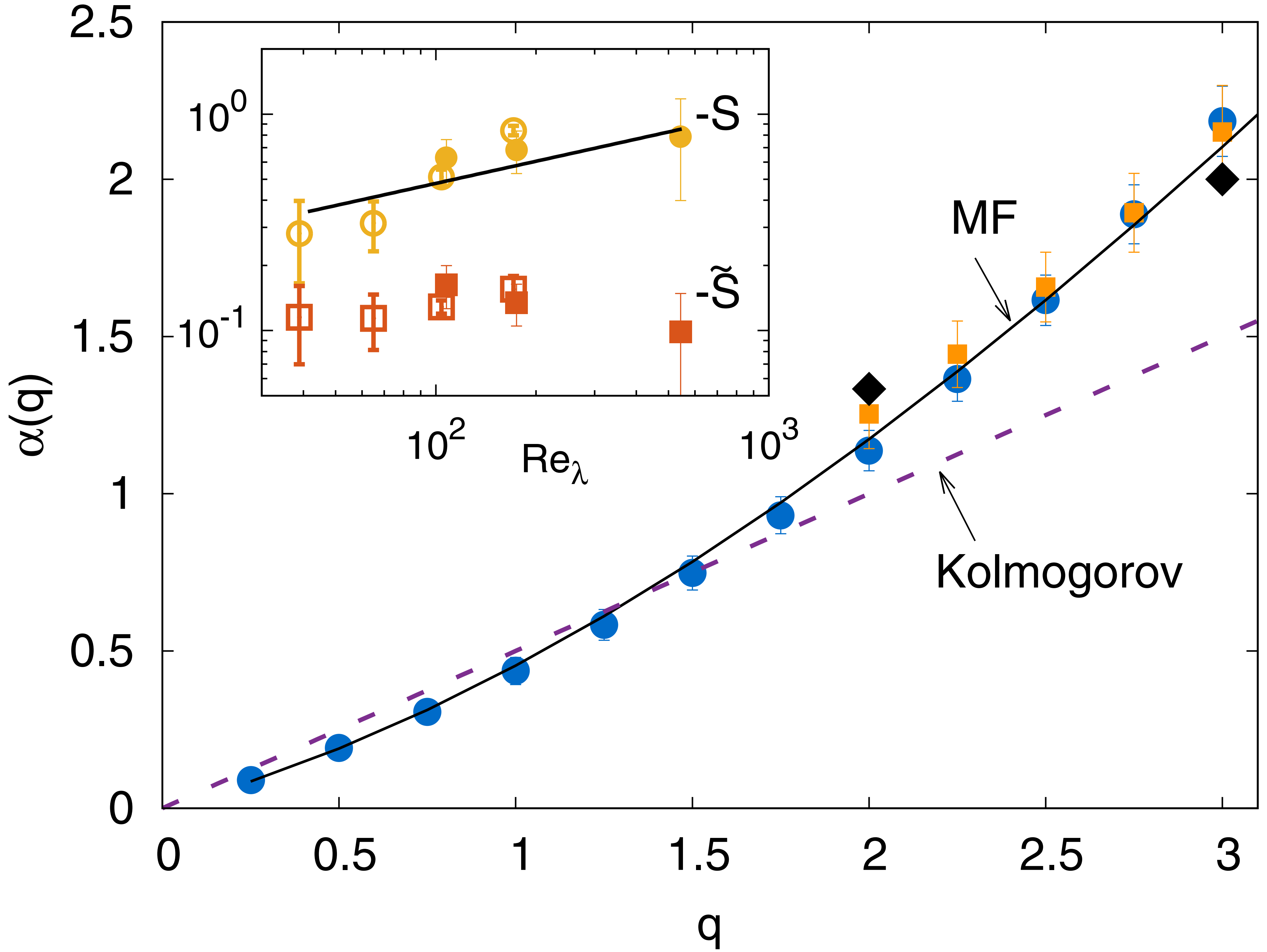
Set	N	Re_λ	ϵ	U	L	T_L	η	τ_η	T
DNS1	2048	544	1.43	1.62	4.51	2.77	0.0021	0.015	15
DNS1	512	176	1.68	1.74	4.70	2.70	0.0083	0.035	10
DNS1	256	115	1.19	1.50	4.26	2.84	0.019	0.066	48
DNS2	1024	171	0.1	0.529	2.22	4.19	0.005	0.063	27
DNS2	512	104	0.1	0.520	2.11	4.06	0.01	0.10	96
DNS2	256	65	0.1	0.513	2.05	3.98	0.02	0.16	165
DNS2	128	38.9	0.1	0.507	1.95	3.85	0.04	0.25	165

$S = \langle p^3 \rangle / \langle p^2 \rangle^{3/2}$

$\tilde{S} = \langle p^3 \rangle / \langle |p|^3 \rangle$

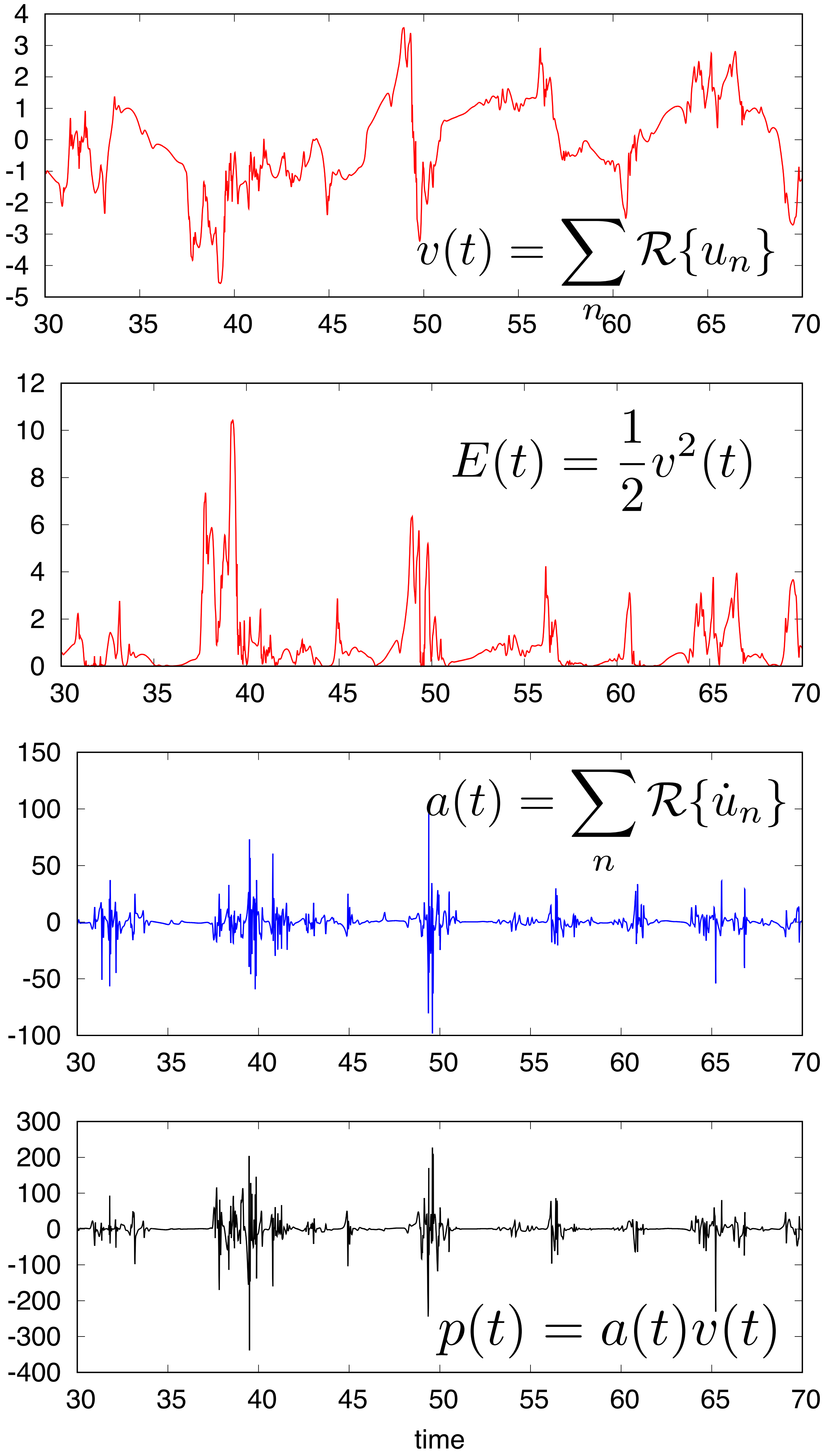
in general

$\mathcal{A}_q / \mathcal{S}_q \approx const$



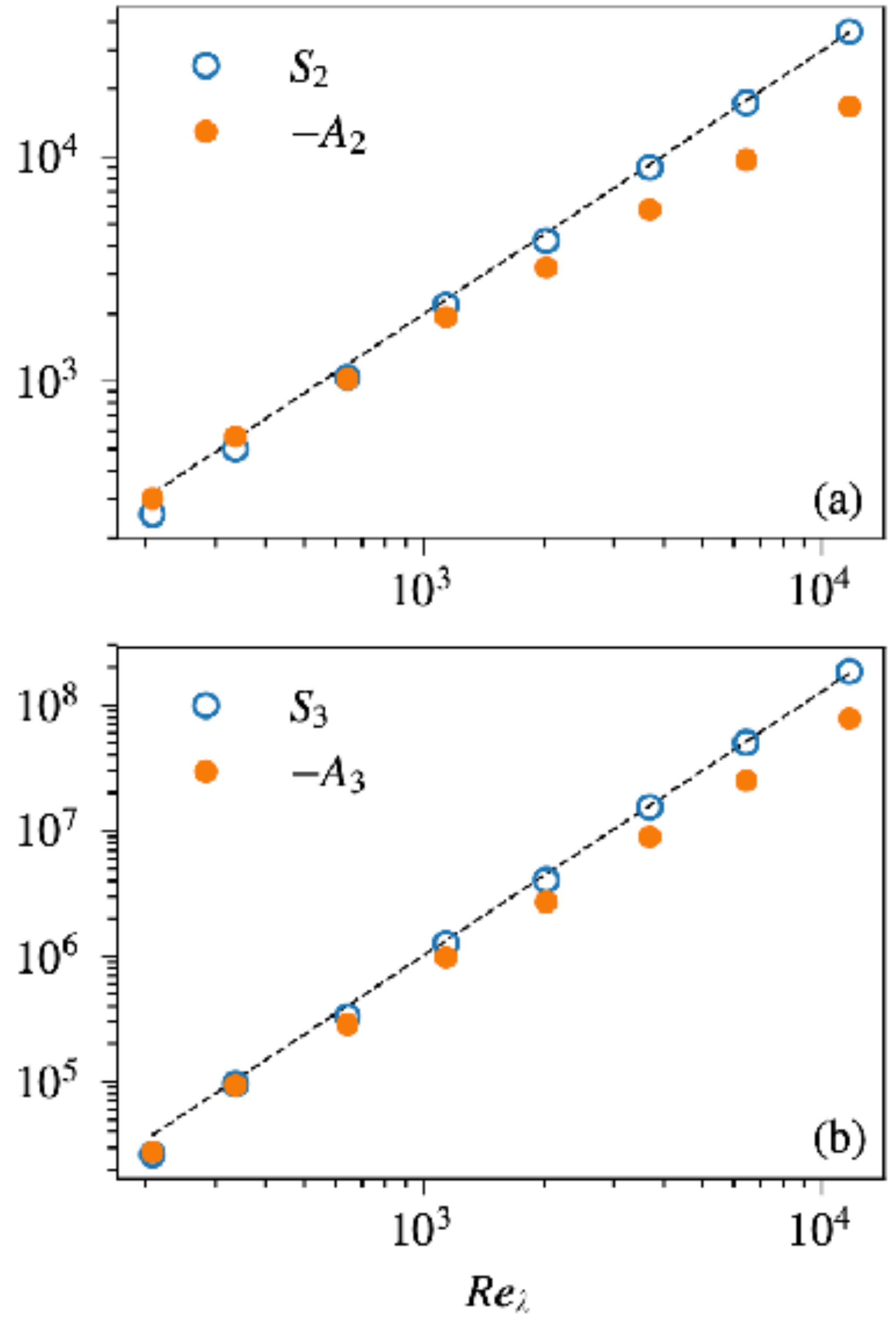
Lagrangian power statistics in SM

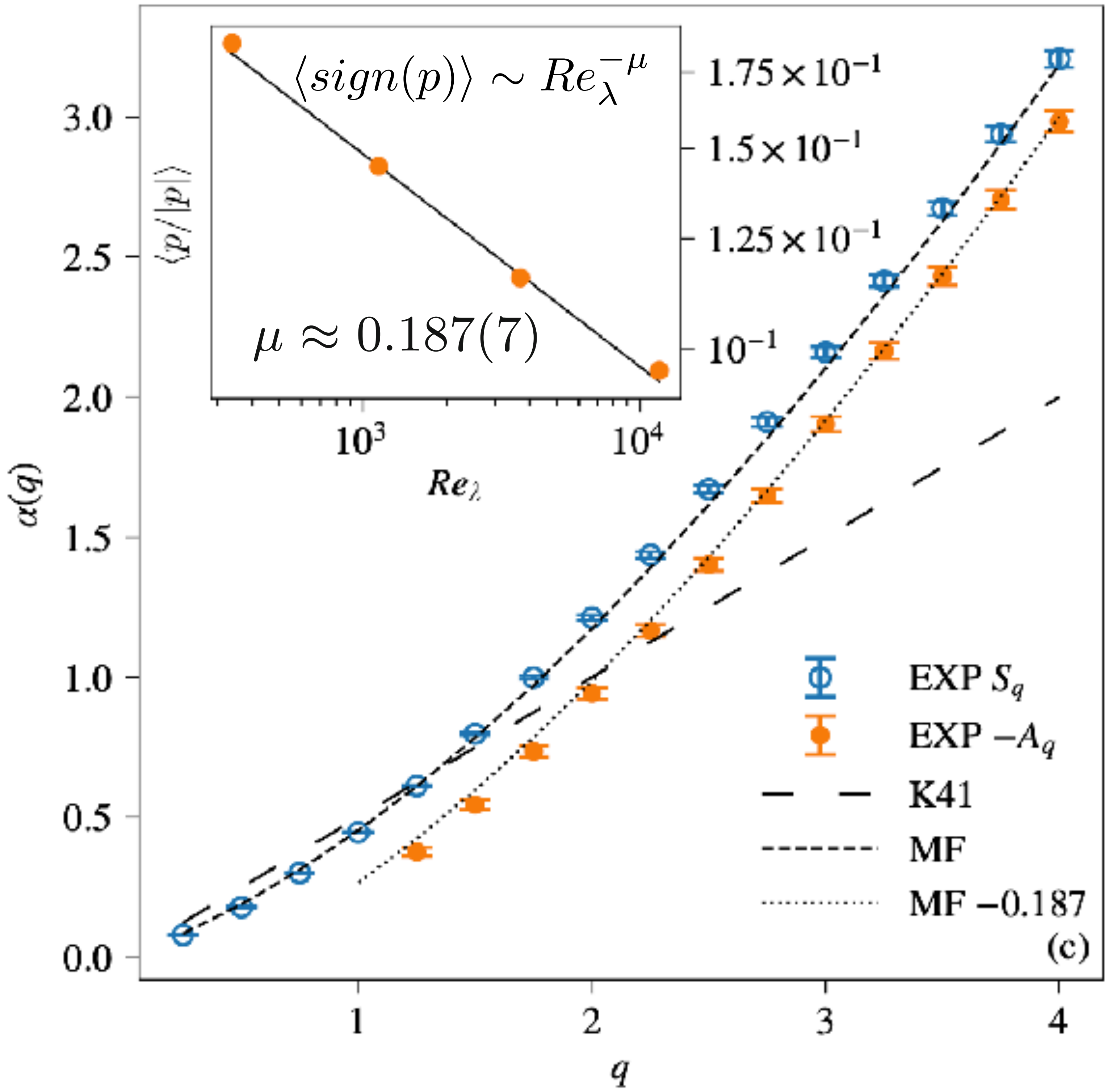
$$(d/dt + \nu k_n^2)u_n = i\left(k_n u_{n+2} u_{n+1}^* - b k_{n-1} u_{n+1} u_{n-1}^* - c k_{n-2} u_{n-1} u_{n-2}\right) + f_n.$$



$$\mathcal{S}_q = \langle |p|^q \rangle / \epsilon^q$$

$$\mathcal{A}_q = \langle p|p|^{q-1} \rangle / \epsilon^q$$





Symmetric moments are well described by MF with same exponents as NS
Antisymmetric are negative but, unlike, NS scales differently from symmetric ones
A deviation that can be accounted for considering the sign statistics

$$-\mathcal{A}_q \sim \mathcal{S}_q \langle sign(p) \rangle \sim Re_\lambda^{\alpha_q - \mu}$$

μ cancellation exponent

(E. Ott, Y. Du, K. Sreenivasan, A. Juneja, A. Suri, PRL 69(18), 2654 (1992))